coordinates, f becomes simply

$$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2,$$

where $\lambda_1, \ldots, \lambda_n$ are the (*necessarily real*) eigenvalues of A. It is thus clear that f is positive definite precisely when all the λ's are positive.

For further information on this topic, see Chapter 10 of Vol. 1 of the book by Gantmacher listed at the end of the chapter.

For a general function $F(x_1, \ldots, x_n)$, with critical point at $(x_1^0, \ldots, x_n^0)$, the method of Section 2.19 leads us to the quadratic form

$$\sum_{i,j} \frac{\partial^2 F}{\partial x_i\, \partial x_j} u_i u_j,$$

where $(u_1, \ldots, u_n)$ is a unit vector and all derivatives are evaluated at the critical point. If this form is positive definite, then F has a minimum at the critical point. Hence *if all eigenvalues of the matrix* $(\partial^2 F/\partial x_i\, \partial x_j)$ *are positive, there is a minimum. Similarly, if all eigenvalues are negative, then there is a maximum.* This matrix is called the Hessian matrix of F.

PROBLEMS

1. Locate the critical points of the following functions, classify them, and graph the functions:

a) $y = x^3 - 3x$, **b)** $y = 2\sin x + \sin 2x$, **c)** $y = e^{-x} - e^{-2x}$.

2. Determine the nature of the critical point of $y = x^n$ $(n = 2, 3, \ldots)$ at $x = 0$.

3. Determine the absolute maximum and absolute minimum, if they exist, of the following functions:

a) $y = \cos x, \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$,

b) $y = \log x, \; 0 < x \le 1$,

c) $y = \tanh x$, all x,

d) $y = \frac{x}{1+x^2}$, all x.

4. Find the critical points of the following functions and test for maxima and minima:

a) $z = \sqrt{1 - x^2 - y^2}$,

b) $z = 1 + x^2 + y^2$,

c) $z = 2x^2 - xy - 3y^2 - 3x + 7y$,

d) $z = x^2 - 5xy - y^2$,

e) $z = x^2 - 2xy + y^2$,

f) $z = x^3 - 3xy^2 + y^3$,

g) $z = x^2 - 2x(\sin y + \cos y) + 1$,

h) $z = xy^2 + x^2y - xy$,

i) $z = x^3 + y^3$,

j) $z = x^4 + 3x^2y^2 + y^4$,

k) $z = [x^2 + (y+1)^2][x^2 + (y-1)^2]$ (interpret geometrically).

5. Find the critical points of the following functions, classify, and graph the level curves of the functions:

a) $z = e^{-x^2-y^2}$,

b) $z = x^4 - y^4$,

c) $z = \sin x \cosh y$,

d) $z = \frac{x}{x^2+y^2}$,

e) $z = x^2 - xy + y^2$,

f) $z = x + y + \sqrt{1 - x^2 - y^2}$.

6. Find the critical points of the following functions with given side conditions and test for maxima and minima:

a) $z = 3x + 4y$, where $x^2 + y^2 = 1$,

b) $z = x^2 + y^2$, where $x^4 + y^4 = 1$,

c) $z = x^2 + 24xy + 8y^2$, where $x^2 + y^2 = 25$,

d) $w = x + z$, where $x^2 + y^2 + z^2 = 1$,

e) $w = xyz$, where $x^2 + y^2 = 1$ and $x - z = 0$,

f) $w = x^2 + y^2 + z^2$, where $x + y + z = 1$ and $x^2 + y^2 - z^2 = 0$.

7. Find the point of the curve

$$x^2 - xy + y^2 - z^2 = 1, \quad x^2 + y^2 = 1$$

nearest to the origin $(0, 0, 0)$.

8. Find the absolute minimum and maximum, if they exist, of the following functions:

a) $z = \frac{1}{1+x^2+y^2}$, all (x, y)

b) $z = xy$, $x^2 + y^2 \le 1$,

c) $w = x + y + z$, $x^2 + y^2 + z^2 \le 1$,

d) $w = e^{-x^2-y^2-z^2}$, all (x, y, z).

9. Determine whether the given quadratic form is positive definite:

a) $3x^2 + 2xy + y^2$,

b) $x^2 - xy - 2y^2$,

c) $\frac{5}{3}x_1^2 + \frac{4}{3}x_1x_2 + 2x_2^2 + \frac{4}{3}x_2x_3 + \frac{7}{3}x_3^2$.

10. Prove the validity of the criterion (2.143) for a minimum, under the conditions stated. [Hint: The function $\nabla_\alpha \nabla_\alpha f(x_0, y_0)$ is continuous in α for $0 \le \alpha \le 2\pi$ and has a minimum M_1 in this interval; by (2.143), $M_1 > 0$. By the Fundamental Lemma of Section 2.6, $\partial z/\partial x$ and $\partial z/\partial y$ have differentials at (x_0, y_0). Show that this implies that

$$\nabla_\alpha f(x, y) = \nabla_\alpha f(x_0, y_0) + s\nabla_\alpha \nabla_\alpha f(x_0, y_0) + \epsilon s = s\nabla_\alpha \nabla_\alpha f(x_0, y_0) + \epsilon s,$$

where $x = x_0 + s\cos\alpha$, $y = y_0 + s\sin\alpha$ ($s > 0$) and $|\epsilon|$ can be made as small as desired by choosing s sufficiently small. Choose δ so that $|\epsilon| < \frac{1}{2}M_1$ for $0 < s < \delta$ and show that

$$\nabla_\alpha f(x, y) = s[\nabla_\alpha \nabla_\alpha f(x_0, y_0) + \epsilon] > 0 \quad \text{for } 0 < s < \delta.$$

Accordingly, f increases steadily, as one recedes from (x_0, y_0) on a straight line in the neighborhood of radius δ of (x_0, y_0).]