graph in an appropriate set E: $|x - x_0| < \delta$, $|y - y_0| < \delta$, $|z - z_0| < \eta$. For the equation $G(x, y_0, z) = 0$ we apply the theorem for the equation $F(x, y) = 0$ to show that $f(x, y_0)$ has a continuous derivative $f_x(x, y_0)$ and the equation

$$f_x(x, y) = -G_x(x, y, f(x, y))/G_z(x, y, f(x, y))$$

is satisfied at (x_0, y_0). By applying the same reasoning to each point $(x_1, y_1, f(x_1, y_1))$ on the graph of f, we conclude that the last equation holds at each point (x, y) with $|x - x_0| < \delta$, $|y - y_0| < \delta$. Since f is continuous, the right hand side of the equation is continuous for $|x - x_0| < \delta$, $|y - y_0| < \delta$ and therefore so also is $f_x(x, y)$. Similarly f_y is continuous and satisfies the analogous equation with G_y replacing G_x.

A similar reasoning applies to the single equation $F(y, x_1, \ldots, x_n) = 0$, to yield a solution $y = f(x_1, \ldots, x_n)$. For the general case of m equations $F_i(y_1, \ldots, y_m, x_1, \ldots, x_n) = 0$, $i = 1, \ldots, m$, one can proceed by induction. The case $m = 1$ has been disposed of. If we have proved the theorem for $m - 1$ equations, then we prove it for m equations by observing that the hypothesis on the Jacobian $\partial(F_1, \ldots, F_m)/\partial(y_1, \ldots, y_m)$ implies that for some i, $\partial F_i/\partial y_m \neq 0$ at P_0. For proper numbering, we can take $i = m$ and then solve the mth equation for y_m (by applying the theorem for a single equation); we substitute the resulting function in the previous equations; one verifies that the new equations in $y_1, \ldots, y_{m-1}, x_1, \ldots, x_n$ still satisfy the appropriate hypotheses. Hence by the induction assumption they yield desired functions $y_i = f_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, m - 1$. Finally,

$$\begin{aligned} y_m &= y_m(y_1, \ldots, y_{m-1}, \; x_1, \ldots, x_n) \\ &= y_m(f_1, \ldots, f_{m-1}, \; x_1, \ldots, x_n) = f_m(x_1, \ldots, x_n). \end{aligned}$$

PROBLEMS

1. Find $(\partial z/\partial x)_y$ and $(\partial z/\partial y)_x$ by Eq. (2.71):
   a) $2x^2 + y^2 - z^2 = 3$
   b) $xyz + 2x^2z + 3xz^2 = 1$
   c) $z^3 + xz + 2yz - 1 = 0$
   d) $e^{xz} + e^{yz} + z - 1 = 0$
2. Given that

$$2x + y - 3z - 2u = 0, \qquad x + 2y + z + u = 0,$$

   find the following partial derivatives:

$$\left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial x}\right)_u, \quad \left(\frac{\partial z}{\partial u}\right)_x, \quad \left(\frac{\partial y}{\partial z}\right)_x.$$

3. Find $(\partial u/\partial x)_y$ and $(\partial u/\partial y)_x$:
   a) $x^2 - y^2 + u^2 + 2v^2 = 1$, $x^2 + y^2 - u^2 - v^2 = 2$
   b) $e^u + xu - yv - 1 = 0$, $e^v - xv + yu - 2 = 0$
   c) $x^2 + xu - yv^2 + uv = 1$, $xu - 2yv = 1$

4. Given that

$$x^2 + y^2 + z^2 - u^2 + v^2 = 1, \qquad x^2 - y^2 + z^2 + u^2 + 2v^2 = 21,$$

a) find du and dv in terms of dx, dy, and dz at the point $x = 1$, $y = 1$, $z = 2$, $u = 3$, $v = 2$;

b) find $(\partial u/\partial x)_{y,z}$ and $(\partial v/\partial y)_{x,z}$ at this point;

c) find approximately the values of u and v for $x = 1.1$, $y = 1.2$, $z = 1.8$.

5. If $xy + 2xu + 3xv + uv - 1 = 0$, $2xy + 3yu - 2xv + 2uv + 2 = 0$, find the Jacobian matrix $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$.

6. Let equations $F_1(x_1, x_2, x_3, x_4) = 0$, $F_2(x_1, x_2, x_3, x_4) = 0$ be given. At a certain point where the equations are satisfied, it is known that

$$\left(\frac{\partial F_i}{\partial x_j}\right) = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 5 & 1 & -1 & 4 \end{bmatrix}.$$

a) Evaluate $(\partial x_1/\partial x_3)_{x_4}$ and $(\partial x_1/\partial x_4)_{x_3}$ at the point.

b) Evaluate $(\partial x_1/\partial x_3)_{x_2}$ and $(\partial x_4/\partial x_3)_{x_2}$ at the point.

c) Evaluate $\partial(x_1, x_2)/\partial(x_3, x_4)$ and $\partial(x_3, x_4)/\partial(x_1, x_2)$ at the point.

7. Prove: If $F(x, y, z) = 0$, then $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x = -1$.

8. In *thermodynamics* the variables p (pressure), T (temperature), U (internal energy), and V (volume) occur. For each substance these are related by two equations, so that any two of the four variables can be chosen as independent, the other two then being dependent. In addition, the second law of thermodynamics implies the relation

a) $\frac{\partial U}{\partial V} - T\frac{\partial p}{\partial T} + p = 0$,
when V and T are independent. Show that this relation can be written in each of the following forms:

b) $\frac{\partial T}{\partial V} + T\frac{\partial p}{\partial U} - p\frac{\partial T}{\partial U} = 0 \quad (U, V \text{ indep.})$,

c) $T - p\frac{\partial T}{\partial p} + \frac{\partial(T, U)}{\partial(V, p)} = 0 \quad (V, p \text{ indep.})$,

d) $\frac{\partial U}{\partial p} + T\frac{\partial V}{\partial T} + p\frac{\partial V}{\partial p} = 0 \quad (p, T \text{ indep.})$,

e) $\frac{\partial T}{\partial p} - T\frac{\partial V}{\partial U} + p\frac{\partial(V, T)}{\partial(U, p)} = 0 \quad (U, p \text{ indep.})$,

f) $T\frac{\partial(p, V)}{\partial(T, U)} - p\frac{\partial V}{\partial U} - 1 = 0 \quad (T, U \text{ indep.})$.

[Hint: The relation (a) implies that if

$$dU = a\,dV + b\,dT, \qquad dp = c\,dV + e\,dT$$

are the expressions for dU and dp in terms of dV and dT, then $a - Te + p = 0$. To prove (b), for example, one assumes relations

$$dT = \alpha\,dV + \beta\,dU, \qquad dp = \gamma\,dV + \delta\,dU.$$

If these are solved for dU and dp in terms of dV and dT, then one obtains expressions for a and e in terms of $\alpha, \beta, \gamma, \delta$. If these expressions are substituted in the equation