in that domain. The equation satisfied by f:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{3.37}$$

is called *Laplace's equation* (see Sections 2.15 and 2.17).

Curl of a curl. Here an expansion into components yields the relation:

$$\text{curl curl } \mathbf{u} = \text{grad div } \mathbf{u} - (\nabla^2 u_x \mathbf{i} + \nabla^2 u_y \mathbf{j} + \nabla^2 u_z \mathbf{k}). \tag{3.38}$$

If one defines the Laplacian of a vector $\mathbf{u}$ to be the vector

$$\nabla^2 \mathbf{u} = \nabla^2 u_x \mathbf{i} + \nabla^2 u_y \mathbf{j} + \nabla^2 u_z \mathbf{k}, \tag{3.39}$$

then (3.38) becomes

$$\text{curl curl } \mathbf{u} = \text{grad div } \mathbf{u} - \nabla^2 \mathbf{u}. \tag{3.40}$$

This identity can be written as an expression for the *gradient of a divergence*:

$$\text{grad div } \mathbf{u} = \text{curl curl } \mathbf{u} + \nabla^2 \mathbf{u}. \tag{3.41}$$

The identities listed here, together with those previously obtained, cover all of interest except for those for *gradient of a scalar product* and *curl of a vector product*; these two are considered in Problems 13 and 14.

PROBLEMS

1. Prove (3.21) and (3.22).
2. Prove that the continuity equation (3.17) can be written in the form

$$\frac{\partial \rho}{\partial t} + \text{grad } \rho \cdot \mathbf{u} + \rho \text{ div } \mathbf{u} = 0$$

or, in terms of the Stokes derivative (Problem 12 following Section 2.8), thus:

$$\frac{D\rho}{Dt} + \rho \text{ div } \mathbf{u} = 0.$$

Prove that (3.17) reduces to (3.18) when $\rho \equiv$ const. It will be shown in Chapter 5 that the same simplification can be made when ρ is variable, provided that the fluid is incompressible. This follows from the fact that $D\rho/Dt$ measures the variation in density at a point moving with the fluid; for an incompressible fluid this local density cannot vary.

3. Prove (3.27) and (3.28).
4. Prove (3.31). Verify by applying to $f = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$.
5. For the given vector field $\mathbf{v}$, verify that curl $\mathbf{v} = \mathbf{0}$ and find all functions f such that grad $f = \mathbf{v}$.
   a) $\mathbf{v} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$.
   b) $\mathbf{v} = e^{xy}[(2y^2 + yz^2)\mathbf{i} + (2xy + xz^2 + 2)\mathbf{j} + 2z\mathbf{k}]$.
6. Prove (3.33). Verify by applying to $\mathbf{v} = x^2yz\mathbf{i} - x^3y^3\mathbf{j} + xyz^2\mathbf{k}$.

7. **a)** Given the vector field $\mathbf{v} = 2x\mathbf{i} + y\mathbf{j} - 3z\mathbf{k}$, verify that div $\mathbf{v} = 0$. Find all vectors $\mathbf{u}$ such that curl $\mathbf{u} = \mathbf{v}$. [Hint: First remark that on the basis of (3.32), all solutions of the equation curl $\mathbf{u} = \mathbf{v}$ are given by $\mathbf{u} = \mathbf{u}_0 + \text{grad } f$, where f is an arbitrary scalar and $\mathbf{u}_0$ is any one vector whose curl is $\mathbf{v}$. To find $\mathbf{u}_0$, assume $\mathbf{u}_0 \cdot \mathbf{k} = 0$.]

b) Proceed as in (a) for $\mathbf{v} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

8. Prove (3.36). Verify that the function f of Problem 4 is harmonic in space (except at the origin). [This function, which represents the electrostatic potential of a charge of $+1$ at the origin, is in a sense the fundamental harmonic function in space, for every harmonic function in space can be represented as a sum, or limit of a sum, of such functions.]

9. Prove (3.35).

10. Prove (3.38).

11. Prove the following identities:

a) div $[\mathbf{u} \times (\mathbf{v} \times \mathbf{w})] = (\mathbf{u} \cdot \mathbf{w})$ div $\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})$ div $\mathbf{w}$ + grad $(\mathbf{u} \cdot \mathbf{w}) \cdot \mathbf{v}$ − grad $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$,

b) div (grad $f \times f$ grad $g) = 0$,

c) curl (curl $\mathbf{v}$ + grad f) = curl curl $\mathbf{v}$,

d) $\nabla^2 f$ = div (curl $\mathbf{v}$ + grad f).

These should be established by means of the identities already found in this chapter and not by expanding into components.

12. One defines the scalar product $\mathbf{u} \cdot \nabla$, with $\mathbf{u}$ on the *left* of the operator ∇, as the operator

$$\mathbf{u} \cdot \nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}.$$

This is thus quite unrelated to $\nabla \cdot \mathbf{u}$ = div $\mathbf{u}$. The operator $\mathbf{u} \cdot \nabla$ can be applied to a scalar f:

$$(\mathbf{u} \cdot \nabla) f = u_x \frac{\partial f}{\partial x} + u_y \frac{\partial f}{\partial y} + u_z \frac{\partial f}{\partial z} = \mathbf{u} \cdot (\nabla f);$$

thus an associative law holds. The operator $\mathbf{u} \cdot \nabla$ can also be applied to a vector $\mathbf{v}$:

$$(\mathbf{u} \cdot \nabla)\,\mathbf{v} = u_x \frac{\partial \mathbf{v}}{\partial x} + u_y \frac{\partial \mathbf{v}}{\partial y} + u_z \frac{\partial \mathbf{v}}{\partial z},$$

where the partial derivatives $\partial \mathbf{v}/\partial x, \ldots$ are defined just as is $d\mathbf{r}/dt$ in Section 2.13; thus one has

$$\frac{\partial \mathbf{v}}{\partial x} = \frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial x}\mathbf{j} + \frac{\partial v_z}{\partial x}\mathbf{k}.$$

a) Show that if $\mathbf{u}$ is a unit vector, then $(\mathbf{u} \cdot \nabla) f = \nabla_{\mathbf{u}} f$.

b) Evaluate $[(\mathbf{i} - \mathbf{j}) \cdot \nabla] f$.

c) Evaluate $[(x\mathbf{i} - y\mathbf{j}) \cdot \nabla](x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k})$.

13. Prove the identity (cf. Problem 12):

$$\text{grad } (\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \times \text{curl } \mathbf{v}) + (\mathbf{v} \times \text{curl } \mathbf{u}).$$

14. Prove the identity (cf. Problem 12):

$$\text{curl } (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \text{ div } \mathbf{v} - \mathbf{v} \text{ div } \mathbf{u} + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}.$$

15. Let $\mathbf{n}$ be the unit outer normal vector to the sphere $x^2 + y^2 + z^2 = 9$ and let $\mathbf{u}$ be the vector $(x^2 - z^2)(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. Evaluate $\partial/\partial n$ (div $\mathbf{u}$) at $(2, 2, 1)$.

16. A rigid body is rotating about the z-axis with angular velocity ω. Show that a typical particle of the body follows a path

$$\overrightarrow{OP} = r\cos(\omega t + \alpha)\mathbf{i} + r\sin(\omega t + \alpha)\mathbf{j} + z\mathbf{k},$$