For reasons to be explained, Δx and Δy can be replaced by dx and dy in (2.20). Thus one has

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy. \tag{2.24}$$

which is the customary way of writing the differential.

The preceding analysis extends at once to functions of three or more variables. For example, if $w = f(x, y, u, v)$, then

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial u}\,du + \frac{\partial w}{\partial v}\,dv. \tag{2.25}$$

EXAMPLE 1 If $z = x^2 - y^2$, then $dz = 2x\,dx - 2y\,dy$. ●

EXAMPLE 2 If $w = \frac{xy}{z}$, then $dw = \frac{y}{z}\,dx + \frac{x}{z}\,dy - \frac{xy}{z^2}\,dz$. ●

PROBLEMS

1. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

a) $z = \frac{y}{x^2 + y^2}$

b) $z = y \sin xy$

c) $x^3 + x^2y - x^2z + z^3 - 2 = 0$

d) $z = \sqrt{e^{x+2y} - y^2}$

e) $z = (x^2 + y^2)^{3/2}$

f) $z = \arcsin(x + 2y)$

g) $e^x + 2e^y - e^z - z = 0$

h) $xy^2 + yz^2 + xyz = 1$

2. A certain function $f(x, y)$ is known to have the following values: $f(0, 0) = 0$, $f(1, 0) = 1$, $f(2, 0) = 4$, $f(0, 1) = -2$, $f(1, 1) = -1$, $f(2, 1) = 2$, $f(0, 2) = -4$, $f(1, 2) = -3$, $f(2, 2) = 0$. Compute approximately the derivatives $f_x(1, 1)$ and $f_y(1, 1)$.

3. Evaluate the indicated partial derivatives:

a) $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial v}{\partial y}\right)_x$ if $u = x^2 - y^2$, $v = x - 2y$

b) $\left(\frac{\partial x}{\partial u}\right)_v$ and $\left(\frac{\partial y}{\partial v}\right)_u$ if $x = e^u \cos v$, $y = e^u \sin v$

c) $\left(\frac{\partial x}{\partial u}\right)_y$ and $\left(\frac{\partial y}{\partial v}\right)_u$ if $u = x - 2y$, $v = u - 2y$

d) $\left(\frac{\partial r}{\partial x}\right)_y$ and $\left(\frac{\partial r}{\partial \theta}\right)_x$ if $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$

4. Find the differentials of the following functions:

a) $z = \frac{x}{y}$

b) $z = \log\sqrt{x^2 + y^2}$

c) $z = \frac{xy}{1 - x - y}$

d) $z = (x - 2y)^5 e^{xy}$

e) $z = \arctan\frac{y}{x}$

f) $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

5. For the given function $z = f(x, y)$, find Δz and dz in terms of Δx and Δy at $x = 1$, $y = 1$. Compare these two functions for selected values of Δx, Δy near 0.

   a) $z = x^2 + 2xy$ b) $z = \frac{x}{x+y}$

6. A certain function $z = f(x, y)$ is known to have the value $f(1, 2) = 3$ and derivatives $f_x(1, 2) = 2$, $f_y(1, 2) = 5$. Make "reasonable" estimates of $f(1.1, 1.8)$, $f(1.2, 1.8)$, and $f(1.3, 1.8)$.
7. Let $z = f(x, y) = xy/(x^2 + y^2)$ except at $(0, 0)$; let $f(0, 0) = 0$. Show that $\partial z/\partial x$ and $\partial z/\partial y$ exist for all (x, y) and are continuous except at $(0, 0)$. Show by the Fundamental Lemma that z has a differential for $(x, y) \neq (0, 0)$ but not at $(0, 0)$, since f is discontinuous at $(0, 0)$. [It is instructive to graph the level curves of f.]

2.7 Differential of Functions of n Variables ■ The Jacobian Matrix

For a function of n variables

$$y = f(x_1, \dots, x_n) \tag{2.26}$$

the differential is obtained as in Section 2.6:

$$dy = f_{x_1}\, dx_1 + \cdots + f_{x_n}\, dx_n. \tag{2.27}$$

Thus it is a linear function of $dx_1, \dots, dx_n$, whose coefficients $f_{x_1}, \dots, f_{x_n}$ are the partial derivatives of f at the point considered. This linear function is a close approximation to the increment Δy in the sense described in Section 2.6:

$$\begin{aligned}\Delta y &= f(x_1 + dx_1, \dots, x_n + dx_n) - f(x_1, \dots, x_n) \\ &= f_{x_1} dx_1 + \cdots + f_{x_n} dx_n + \epsilon_1 dx_1 + \cdots + \epsilon_n dx_n,\end{aligned} \tag{2.28}$$

where

$$\epsilon_1 \to 0, \dots, \epsilon_n \to 0 \quad \text{as} \quad dx_1 \to 0, \dots, dx_n \to 0.$$

On occasion, one has to deal with several functions of n variables:

$$\begin{cases} y_1 = f_1(x_1, \dots, x_n), \\ \vdots \\ y_m = f_m(x_1, \dots, x_n). \end{cases} \tag{2.29}$$

If these functions have continuous partial derivatives in a domain D of E^n, then all have differentials:

$$\begin{cases} dy_1 = \frac{\partial f_1}{\partial x_1} dx_1 + \cdots + \frac{\partial f_1}{\partial x_n} dx_n, \\ \vdots \\ dy_m = \frac{\partial f_m}{\partial x_1} dx_1 + \cdots + \frac{\partial f_m}{\partial x_n} dx_n. \end{cases} \tag{2.30}$$