Accordingly,

$$\frac{\partial Z}{\partial y_1} - \frac{\partial Y}{\partial z_1} = \int_0^1 \left(t^2 \frac{\partial L}{\partial t} + 2tL \right) dt = \int_0^1 \frac{\partial}{\partial t}(t^2 L)\, dt$$
$$= t^2 L \Big|_{t=0}^{t=1} = L(x_1, y_1, z_1).$$

This gives the first of (5.100). The other two equations are proved in the same way. ●

The solution **v** can be expressed in the compact form:

$$\mathbf{v}(x, y, z) = \int_0^1 t\mathbf{u}(xt, yt, zt) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\, dt. \quad \textbf{(5.103)}$$

If the vector field **u** is *homogeneous of degree n*, that is,

$$\mathbf{u}(xt, yt, zt) = t^n \mathbf{u}(x, y, z)$$

(Problem 11 following Section 2.8), the formula can be simplified further:

$$\mathbf{v} = \int_0^1 t^{n+1} \mathbf{u}(x, y, z) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\, dt$$
$$= \frac{1}{n+2}(\mathbf{u} \times \mathbf{r}), \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

For more on this topic, see pages 487–489 of vol. 58 (1951) and pages 409–442 of vol. 109 (2002) of the *American Mathematical Monthly*.

PROBLEMS

1. Evaluate by Stokes's theorem:
   a) $\int_C u_T\, ds$, where C is the circle $x^2 + y^2 = 1$, $z = 2$, directed so that y increases for positive x, and **u** is the vector $-3y\mathbf{i} + 3x\mathbf{j} + \mathbf{k}$;
   b) $\int_C 2xy^2 z\, dx + 2x^2 yz\, dy + (x^2 y^2 - 2z)\, dz$ around the curve $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \le t \le 2\pi$, directed with increasing t.
2. By showing that the integrand is an exact differential, evaluate
   a) $\int_{(1,1,2)}^{(3,5,0)} yz\, dx + xz\, dy + xy\, dz$ on any path;
   b) $\int_{(1,0,0)}^{(1,0,2\pi)} \sin yz\, dx + xz \cos yz\, dy + xy \cos yz\, dz$ on the helix $x = \cos t$, $y = \sin t$, $z = t$.
3. Let C be a simple closed *plane* curve in space. Let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a unit vector normal to the plane of C and let the direction on C match that of **n**. Prove that

$$\frac{1}{2} \int_C (bz - cy)\, dx + (cx - az)\, dy + (ay - bx)\, dz$$

   equals the plane area enclosed by C. What does the integral reduce to when C is in the xy-plane?
4. Let $\mathbf{u} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\mathbf{k}$ and let D be the interior of the torus obtained by rotating the circle $(x - 2)^2 + z^2 = 1$, $y = 0$ about the z-axis. Show that curl $\mathbf{u} = \mathbf{0}$ in D but

$\int_C u_T\, ds$ is not zero when C is the circle $x^2 + y^2 = 4$, $z = 0$. Determine the possible values of the integral $\int_{(2,0,0)}^{(0,2,0)} u_T\, ds$ on a path in D.

5. **a)** Show that if $\mathbf{v}$ is one solution of the equation curl $\mathbf{v} = \mathbf{u}$ for given $\mathbf{u}$ in a simply connected domain D, then all solutions are given by $\mathbf{v}$ + grad f, where f is an arbitrary differentiable scalar in D.

b) Find all vectors $\mathbf{v}$ such that curl $\mathbf{v} = \mathbf{u}$ if

$$\mathbf{u} = (2xyz^2 + xy^3)\mathbf{i} + (x^2y^2 - y^2z^2)\mathbf{j} - (y^3z + 2x^2yz)\mathbf{k}.$$

6. Show that if f and g are scalars having continuous second partial derivatives in a domain D, then

$$\mathbf{u} = \nabla f \times \nabla g$$

is solenoidal in D. (It can be shown that every solenoidal vector has such a representation, at least in a suitably restricted domain.)

7. Show that if $\iint_S u_n\, d\sigma = 0$ for every oriented spherical surface S in a domain D and the components of $\mathbf{u}$ have continuous derivatives in D, then $\mathbf{u}$ is solenoidal in D. Does the converse hold?

8. Let C and S be as in Stokes's theorem. Prove, under appropriate assumptions:

a) $\int_C f\mathbf{T} \cdot \mathbf{i}\, ds = \iint_S \mathbf{n} \times \nabla f \cdot \mathbf{i}\, d\sigma$;
[Hint: Apply Stokes's theorem, taking $\mathbf{u} = f\mathbf{i}$. Evaluate curl $\mathbf{u}$ by (3.28).]

b) $\int_C f\mathbf{T}\, ds = \iint_S \mathbf{n} \times \nabla f\, d\sigma$.
[Hint: These are vector integrals, as in Section 4.5. Show by (a) that the x-components of both sides are equal and, similarly, that the y- and z-components are equal.]

9. The operator $\mathbf{v} \times \nabla$ is defined formally as $(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \times (\nabla_x\mathbf{i} + \nabla_y\mathbf{j} + \nabla_z\mathbf{k}) = (v_y\nabla_z - v_z\nabla_y)\mathbf{i} + \cdots$
Show that, formally:

a) $(\mathbf{v} \times \nabla) \cdot \mathbf{u} = \mathbf{v} \cdot \nabla \times \mathbf{u} = \mathbf{v} \cdot \text{curl}\, \mathbf{u}$;

b) $(\mathbf{v} \times \nabla) \times \mathbf{u} = \nabla_u(\mathbf{v} \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u})\mathbf{v}$, where $\nabla_u(\mathbf{v} \cdot \mathbf{u})$ indicates that $\mathbf{v}$ is treated as constant: $\nabla_u(\mathbf{v} \cdot \mathbf{u}) = v_x\nabla u_x + v_y\nabla u_y + v_z\nabla u_z$.

10. Let C and S be as in Stokes's theorem. Show with the aid of Problem 9:

a) $\int_C \mathbf{T} \times \mathbf{u} \cdot \mathbf{i}\, ds = \iint_S (\mathbf{n} \times \nabla) \times \mathbf{u} \cdot \mathbf{i}\, d\sigma$;

b) $\int_C \mathbf{T} \times \mathbf{u}\, ds = \iint_S (\mathbf{n} \times \nabla) \times \mathbf{u}\, d\sigma$.

*5.14 Change of Variables in a Multiple Integral

The formula for change of variables in a double integral:

$$\iint\limits_{R_{xy}} F(x, y)\, dx\, dy = \iint\limits_{R_{uv}} F[f(u, v), g(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv, \qquad \textbf{(5.104)}$$

is given in Section 4.6. In this section we shall give a proof of this formula under appropriate assumptions. We shall also indicate how widely the formula is applicable and shall explain the more general formula:

$$\delta \iint\limits_{R_{xy}} F(x, y)\, dx\, dy = \iint\limits_{R_{uv}} F[f(u, v), g(u, v)] \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv, \qquad \textbf{(5.105)}$$