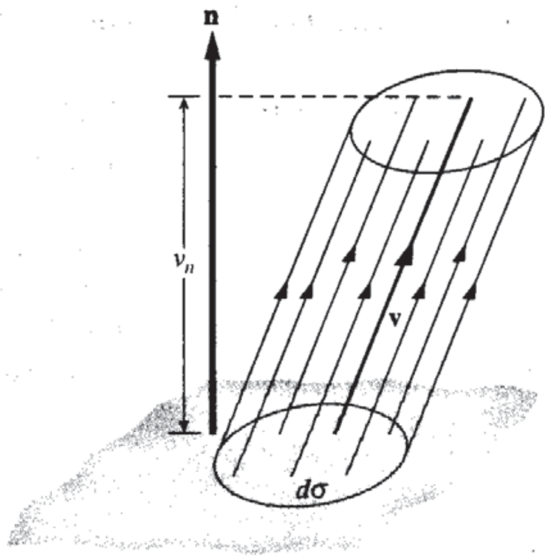

Figure 5.30 Flux $= \iint v_n \, d\sigma$.

mass per unit volume, by (5.91). But this is precisely the rate at which the density is *decreasing* at the point (x_1, y_1, z_1). Hence

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \mathbf{v} = -\operatorname{div} (\rho \mathbf{u})$$

or

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{u}) = 0. \tag{5.92}$$

This is the *continuity equation* of hydrodynamics. It expresses the *conservation of mass*. Another derivation is given in Problem 9 following Section 5.15.

PROBLEMS

1. Evaluate by the divergence theorem:

a) $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where S is the sphere $x^2 + y^2 + z^2 = 1$ and $\mathbf{n}$ is the outer normal;

b) $\iint_S v_n \, d\sigma$, where $\mathbf{v} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, $\mathbf{n}$ is the outer normal and S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$;

c) $\iint_S e^y \cos z \, dy\,dz + e^x \sin z \, dz\,dx + e^x \cos y \, dx\,dy$, with S and $\mathbf{n}$ as in (a);

d) $\iint_S \nabla F \cdot \mathbf{n} \, d\sigma$ if $F = x^2 + y^2 + z^2$, $\mathbf{n}$ is the exterior normal, and S bounds a solid region R;

e) $\iint_S \nabla F \cdot \mathbf{n} \, d\sigma$ if $F = 2x^2 - y^2 - z^2$, with $\mathbf{n}$ and S as in (d);

f) $\iint_S \nabla F \cdot \mathbf{n} \, d\sigma$ if $F = [(x-2)^2 + y^2 + z^2]^{-1/2}$ and S and $\mathbf{n}$ are as in (a).

2. Let S be the boundary surface of a region R in space and let $\mathbf{n}$ be its outer normal. Prove the formulas:

a) $V = \iint_S x\,dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy$
$= \frac{1}{3} \iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$,
where V is the volume of R;

b) $\iint_S x^2\,dy\,dz + 2xy\,dz\,dx + 2xz\,dx\,dy = 6V\bar{x}$,
where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of R;

c) $\iint_S \operatorname{curl} \mathbf{v} \cdot \mathbf{n}\,d\sigma = 0$, where $\mathbf{v}$ is an arbitrary vector field.

3. Deduce the results of Problem 9(a), (b), (c) following Section 5.10 by proving (c) first, using the incompressibility of the flow of constant velocity $\mathbf{v}$.

4. Let S be the boundary surface of a region R, with outer normal $\mathbf{n}$, as in the Divergence theorem. Let $f(x, y, z)$ and $g(x, y, z)$ be functions defined and continuous, with continuous first and second derivatives, in a domain D containing R. Prove the following relations:

a) $\iint_S f\,\partial g/\partial n\,d\sigma = \iiint_R f\nabla^2 g\,dx\,dy\,dz + \iiint_R (\nabla f \cdot \nabla g)\,dx\,dy\,dz$;
[Hint: use the identity $\nabla \cdot (f\mathbf{u}) = \nabla f \cdot \mathbf{u} + f(\nabla \cdot \mathbf{u})$.]

b) if g is harmonic in D, then

$$\iint_S \frac{\partial g}{\partial n}\,d\sigma = 0;$$

[Hint: Put $f = 1$ in (a).]

c) if f is harmonic in D, then

$$\iint_S f\frac{\partial f}{\partial n}\,d\sigma = \iiint_R |\nabla f|^2\,dx\,dy\,dz;$$

d) if f is harmonic in D and $f \equiv 0$ on S, then $f \equiv 0$ in R [cf. the last paragraph before the remarks at the end of Section 4.3];

e) if f and g are harmonic in D and $f \equiv g$ on S, then $f \equiv g$ in R; [Hint: Use (d).]

f) if f is harmonic in D and $\partial f/\partial n = 0$ on S, then f is constant in R;

g) if f and g are harmonic in D and $\partial f/\partial n = \partial g/\partial n$ on S, then $f = g + \text{const}$ in R;

h) if f and g are harmonic in R, and

$$\frac{\partial f}{\partial n} = -f + h, \qquad \frac{\partial g}{\partial n} = -g + h \text{ on } S, \qquad h = h(x, y, z),$$

then

$$f \equiv g \text{ in } R;$$

i) if f and g both satisfy the same *Poisson equation* in R,

$$\nabla^2 f = -4\pi h, \qquad \nabla^2 g = -4\pi h, \qquad h = h(x, y, z),$$

and $f = g$ on S, then

$$f \equiv g \text{ in } R;$$

j) $\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)d\sigma = \iiint_R (f\nabla^2 g - g\nabla^2 f)\,dx\,dy\,dz$;
[Hint: Use (a).]

k) if f and g are harmonic in R, then

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)d\sigma = 0;$$

l) if f and g satisfy the equations:

$$\nabla^2 f = hf, \qquad \nabla^2 g = hg, \qquad h = h(x, y, z),$$

in R, then

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) d\sigma = 0.$$

Remark Parts (a) and (j) are known as *Green's first and second identities*, respectively.

5. Let S and R be as in Problem 4. Prove, under appropriate continuity assumptions:

a) $\iint_S f\mathbf{n} \cdot \mathbf{i}\, d\sigma = \iiint_R \frac{\partial f}{\partial x}\, dV.$
[Hint: Apply the Divergence theorem.]

b) $\iint_S f\mathbf{n}\, d\sigma = \iiint_R \nabla f\, dV.$
[Hint: These are integrals of vectors as in Section 4.5. Use (a) to show that the x-components of both sides are equal and, similarly, that the y- and z-components are equal.]

c) $\iint_S \mathbf{v} \times \mathbf{i} \cdot \mathbf{n}\, d\sigma = \iiint_R \operatorname{curl} \mathbf{v} \cdot \mathbf{i}\, dV.$
[Hint: Apply the Divergence theorem and then evaluate div $(\mathbf{v} \times \mathbf{i})$ by (3.35).]

d) $\iint_S \mathbf{n} \times \mathbf{v}\, d\sigma = \iiint_R \operatorname{curl} \mathbf{v}\, dV.$
[Hint: These are vector integrals. Use (c) to show that the x-components of both sides are equal and, similarly, that the y- and z-components are equal.]

5.12 STOKES'S THEOREM

It was seen in Section 5.5 that Green's theorem can be written in the form

$$\oint_C u_T\, ds = \iint_R \operatorname{curl}_z \mathbf{u}\, dx\, dy.$$

This suggests that for any simple closed plane curve C in space (Fig. 5.31),

$$\int_C u_T\, ds = \iint_S \operatorname{curl}_n \mathbf{u}\, d\sigma, \tag{5.93}$$

where $\mathbf{n}$ is normal to the plane in which C lies, S is the planar surface bounded by C, and the direction of C is positive in terms of the orientation of S determined by $\mathbf{n}$.

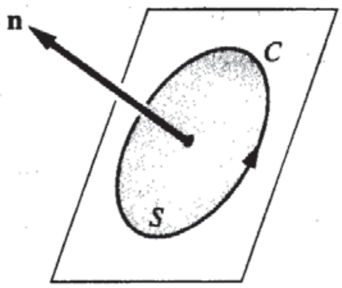

Figure 5.31 Case of Stokes's theorem.