One can also define this as the limit of a sum, as for the line integral. For the surface $z = f(x, y)$, one finds

$$d\boldsymbol{\sigma} = \pm\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dx\, dy.$$

This can also be written:

$$d\boldsymbol{\sigma} = \mathbf{n}|\sec\gamma|\, dx\, dy = \frac{\mathbf{n}}{|\mathbf{n}\cdot\mathbf{k}|}\, dx\, dy.$$

From this and the corresponding expressions for surfaces $x = g(y, z)$, $y = h(x, z)$, one obtains the formulas

$$\iint\limits_S (\mathbf{v}\cdot\mathbf{n})\, d\sigma = \iint\limits_{R_{xy}} \frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}\cdot\mathbf{k}|}\, dx\, dy = \iint\limits_{R_{yz}} \frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}\cdot\mathbf{i}|}\, dy\, dz = \iint\limits_{R_{zx}} \frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}\cdot\mathbf{j}|}\, dz\, dx. \quad \textbf{(5.83)}$$

For the surfaces in parametric form we have

$$d\boldsymbol{\sigma} = \pm\left(\frac{\partial(y, z)}{\partial(u, v)}\mathbf{i} + \frac{\partial(z, x)}{\partial(u, v)}\mathbf{j} + \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}\right) du\, dv$$

or, more concisely, by (5.82),

$$d\boldsymbol{\sigma} = \pm(\mathbf{P}_1 \times \mathbf{P}_2)\, du\, dv.$$

The formal properties of line and surface integrals are analogous to those for line integrals in the plane (Section 5.3) and need no special discussion here. Furthermore, the definitions and properties of line and surface integrals carry over without change to piecewise smooth curves and surfaces, provided that the surfaces are orientable.

PROBLEMS

1. Evaluate the line integrals:

a) $\int_{(1,0,0)\,C}^{(1,0,2\pi)} z\, dx + x\, dy + y\, dz$, where C is the curve $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$;

b) $\int_{(1,0,1)}^{(2,3,2)} x^2\, dx - xz\, dy + y^2\, dz$ on the straight line joining the two points;

c) $\int_{(1,1,0)}^{(0,0,\sqrt{2})} x^2 yz\, ds$ on the curve $x = \cos t$, $y = \cos t$, $z = \sqrt{2}\sin t$, $0 \le t \le \frac{\pi}{2}$;

d) $\int_C u_T\, ds$, where $\mathbf{u} = 2xy^2z\mathbf{i} + 2x^2yz\mathbf{j} + x^2y^2\mathbf{k}$ and C is the circle $x = \cos t$, $y = \sin t$, $z = 2$, directed by increasing t;

e) $\int_C u_T\, ds$, where $\mathbf{u} = \operatorname{curl} \mathbf{v}$, $\mathbf{v} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$, and C is the path $x = 2t + 1$, $y = t^2$, $z = 1 + t^3$, $0 \le t \le 1$, directed by increasing t.

2. If $\mathbf{u} = \operatorname{grad} F$ in a domain D, then show that

a) $\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} u_T\, ds = F(x_2, y_2, z_2) - F(x_1, y_1, z_1)$, where the integral is along any path in D joining the two points;

b) $\int_C u_T\, ds = 0$ on any closed path in D.

3. Let a wire be given as a curve C in space. Let its density (mass per unit length) be $\delta = \delta(x, y, z)$, where (x, y, z) is a variable point in C. Justify the following formulas:

a) length of wire $= \int_C ds = L$;

b) mass of wire $= \int_C \delta\, ds = M$;

c) center of mass of the wire is $(\bar{x}, \bar{y}, \bar{z})$, where

$$M\bar{x} = \int_C x\,\delta\, ds, \qquad M\bar{y} = \int_C y\,\delta\, ds, \qquad M\bar{z} = \int_C z\,\delta\, ds;$$

d) moment of inertia of the wire about the z axis is

$$I_z = \int_C (x^2 + y^2)\,\delta\, ds.$$

4. Formulate and justify the formulas analogous to those of Problem 3 for the surface area, mass, center of mass, and moment of inertia of a thin curved sheet of metal forming a surface S in space.

5. Evaluate the following surface integrals:

a) $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where S is the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1) and the normal points away from (0, 0, 0);

b) $\iint_S dy\,dz + dz\,dx + dx\,dy$, where S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $x^2 + y^2 \le 1$, and the normal is the upper normal;

c) $\iint_S (x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma$ for the surface of part (b);

d) $\iint_S x^2 z\,d\sigma$, where S is the cylindrical surface $x^2 + y^2 = 1$, $0 \le z \le 1$.

6. Evaluate the surface integrals of Problem 5, using the parametric representation:

a) $x = u + v$, $y = u - v$, $z = 1 - 2u$

b) $x = \sin u \cos v$, $y = \sin u \sin v$, $z = \cos u$

c) same as (b)

d) $x = \cos u$, $y = \sin u$, $z = v$

7. Evaluate the surface integrals:

a) $\iint_S \mathbf{w} \cdot \mathbf{n}\,d\sigma$, if $\mathbf{w} = xy^2 z\mathbf{i} - 2x^3\mathbf{j} + yz^2\mathbf{k}$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$, and $\mathbf{n}$ is upper;

b) $\iint_S \mathbf{w} \cdot \mathbf{n}\,d\sigma$, if $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, S is the surface $x = e^u \cos v$, $y = e^u \sin v$, $z = \cos v \sin v$, $0 \le u \le 1$, $0 \le v \le \pi/2$, and $\mathbf{n}$ is given by (5.82) with the + sign;

c) $\iint_S \frac{\partial w}{\partial n}\,d\sigma$ if $w = x^2 y^2 z$ and S and $\mathbf{n}$ are as in (a);

d) $\iint_S \frac{\partial w}{\partial n}\,d\sigma$ if $w = x^2 - y^2 + z^2$ and S and $\mathbf{n}$ are as in (b);

e) $\iint_S \operatorname{curl} \mathbf{u} \cdot \mathbf{n}\,d\sigma$ if $\mathbf{u} = yz\mathbf{i} - xz\mathbf{j} + xz\mathbf{k}$, S is the triangle with vertices (1, 2, 8), (3, 1, 9), (2, 1, 7) and $\mathbf{n}$ is upper.

8. a) Let a surface S: $z = f(x, y)$ be defined by an implicit equation $F(x, y, z) = 0$. Show that the surface integral $\iint H\,d\sigma$ over S becomes

$$\iint_{R_{xy}} \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\,\frac{H}{\left|\frac{\partial F}{\partial z}\right|}\,dx\,dy,$$

provided that $\frac{\partial F}{\partial z} \ne 0$.

b) Prove that for the surface of part (a) with $\mathbf{n} = \nabla F/|\nabla F|$,

$$\iint_S (\mathbf{v}\cdot\mathbf{n})\,d\sigma = \iint_{R_{xy}} (\mathbf{v}\cdot\nabla F)\frac{1}{\left|\frac{\partial F}{\partial z}\right|}\,dx\,dy.$$

c) Prove (5.81).

d) Prove that (5.81) reduces to (5.80) when $x = u$, $y = v$, $z = f(u, v)$.

9. Let S be an oriented surface in space that is planar; that is, S lies in a plane. With S one can associate the vector $\mathbf{S}$, which has the direction of the normal chosen on S and has a length equal to the area of S.

a) Show that if S_1, S_2, S_3, S_4 are the faces of a tetrahedron, oriented so that the normal is the exterior normal, then

$$\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = \mathbf{0}.$$

[Hint: Let $\mathbf{S}_i = A_i\mathbf{n}_i$ ($A_i > 0$) for $i = 1, \ldots, 4$ and let $\mathbf{S}_1 + \cdots + \mathbf{S}_4 = \mathbf{b}$. Let p_1 be the foot of the altitude on face S_1 and join p_1 to the vertices of S_1 to form three triangles of areas $A_{12}, \ldots, A_{14}$. Show that, for proper numbering, $A_{1j} = \pm A_j\mathbf{n}_j\cdot\mathbf{n}_1$, with $+$ or $-$ according as $\mathbf{n}_j\cdot\mathbf{n}_1 > 0$ or < 0, and $A_{1j} = 0$ if $\mathbf{n}_j\cdot\mathbf{n}_1 = 0$ ($j = 2, 3, 4$). Hence deduce that $\mathbf{b}\cdot\mathbf{n}_j = 0$ for $j = 2, 3, 4$ and thus $\mathbf{b}\cdot\mathbf{b} = 0$.]

b) Show that the result of (a) extends to an arbitrary convex polyhedron with faces $S_1, \ldots, S_n$, that is, that

$$\mathbf{S}_1 + \mathbf{S}_2 + \cdots + \mathbf{S}_n = \mathbf{0},$$

when the orientation is that of the exterior normal.

c) Using the result of (b), indicate a reasoning to justify the relation

$$\iint_S \mathbf{v}\cdot d\boldsymbol{\sigma} = 0$$

for any convex closed surface S (such as the surface of a sphere or ellipsoid), provided that $\mathbf{v}$ is a constant vector.

d) Apply the result of (b) to a triangular prism whose edges represent the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}+\mathbf{b}$, $\mathbf{c}$ to prove the *distributive law* (Equation (1.19))

$$\mathbf{c}\times(\mathbf{a}+\mathbf{b}) = \mathbf{c}\times\mathbf{a} + \mathbf{c}\times\mathbf{b}$$

for the vector product. This is the method used by Gibbs (cf. the book by Gibbs listed at the end of this chapter).

5.11 The Divergence Theorem

It was pointed out in Section 5.5 that Green's theorem can be written in the form

$$\int_C v_n\,ds = \iint_R \operatorname{div}\mathbf{v}\,dx\,dy.$$

The following generalization thus appears natural:

$$\iint_S v_n\,d\sigma = \iiint_R \operatorname{div}\mathbf{v}\,dx\,dy\,dz,$$

where S is a surface forming the complete boundary of a bounded closed region R in space and $\mathbf{n}$ is the outer normal of S, that is, the one pointing away from R.