the correspondence must be one-to-one. From the level curves, one can follow the variation of u and v on the boundary of R_{xy} and thereby determine the boundary of R_{uv}. It can be shown that if R_{xy} and R_{uv} are each bounded by a single closed curve, as in Fig. 4.8, if the correspondence between (x, y) and (u, v) is one-to-one on these boundary curves, and $J \neq 0$ in R_{uv}, then the correspondence is necessarily one-to-one in all of R_{xy} and R_{uv}. For a further discussion of this point, the reader is referred to Section 5.14. Actually, the conditions that the correspondence be one-to-one and that $J \neq 0$ are not vital for the theorem. It is shown in Section 5.14 that (4.61) can be written in a different form that covers the more general cases.

PROBLEMS

1. Evaluate with the aid of the substitution indicated:

a) $\int_0^1 (1-x^2)^{3/2}\,dx,\ x = \sin\theta$

b) $\int_0^1 \frac{1}{1+\sqrt{1+x}}\,dx,\ x = u^2 - 1$

c) $\int_0^{\pi/2} \frac{1}{\sin x + \cos x + 2}\,dx,\ t = \tan(x/2)$

d) $\int_0^{\pi/4} \frac{x\cos x(x\sin x - \cos x)}{1 + x\cos x}\,dx,\ t = 1 + x\cos x$

2. Prove the formula

$$\int_{u_1}^{u_2} \phi'(u)\,du = \phi(u_2) - \phi(u_1)$$

as a special case of (4.60).

3. a) Prove that (4.60) remains valid for improper integrals, that is, if $f(x)$ is continuous for $x_1 \leq x < x_2$, $x(u)$ is defined and has a continuous derivative for $u_1 \leq u < u_2$, with $x(u_1) = x_1$, $\lim_{u\to u_2} x(u) = x_2$, and $f[x(u)]$ is continuous for $u_1 \leq u < u_2$. [Hint: Use the fact that (4.60) holds with u_2 and x_2 replaced by u_0 and $x_0 = x(u_0)$, $u_1 < u_0 < u_2$. Then let u_0 approach u_2. One concludes that if either side of the equation has a limit, then the other side has a limit also and the limits are equal. Note that x_2 or u_2 or both may be ∞.]

b) Evaluate $\int_1^\infty \frac{1}{x^2}\sinh\frac{1}{x}\,dx$ by setting $u = \frac{1}{x}$.

c) Evaluate $\int_0^\infty (1 - \tanh x)\,dx$ by setting $u = \tanh x$.

4. Evaluate the following integrals with the aid of the substitution suggested:

a) $\iint_{R_{xy}} (1-x^2-y^2)\,dx\,dy$, where R_{xy} is the region $x^2+y^2 \leq 1$, using $x = r\cos\theta$, $y = r\sin\theta$;

b) $\iint_R \frac{y\sqrt{x^2+y^2}}{x}\,dx\,dy$, where R is the region $1 \leq x \leq 2, 0 \leq y \leq x$, using $x = r\cos\theta$, $y = r\sin\theta$;

c) $\iint_{R_{xy}} (x-y)^2\sin^2(x+y)\,dx\,dy$, where R_{xy} is the parallelogram with successive vertices $(\pi, 0), (2\pi, \pi), (\pi, 2\pi), (0, \pi)$, using $u = x - y$, $v = x + y$;

d) $\iint_R \frac{(x-y)^2}{1+x+y}\,dx\,dy$, where R is the trapezoidal region bounded by the lines $x + y = 1$, $x + y = 2$ in the first quadrant, using $u = 1 + x + y$, $v = x - y$;

e) $\iint_R \sqrt{5x^2+2xy+2y^2}\,dx\,dy$ over the region R bounded by the ellipse $5x^2 + 2xy + 2y^2 = 1$, using $x = u + v$, $y = -2u + v$.

5. Verify that the transformation $u = e^x \cos y$, $v = e^x \sin y$ defines a one-to-one mapping of the rectangle R_{xy}: $0 \le x \le 1$, $0 \le y \le \pi/2$ onto a region of the uv-plane and express as an iterated integral in u, v the integral

$$\iint\limits_{R_{xy}} \frac{e^{2x}}{1 + e^{4x}\cos^2 y \sin^2 y}\, dx\, dy.$$

6. Verify that the transformation

$$u = 2xy, \qquad v = x^2 - y^2$$

defines a one-to-one mapping of the square $0 \le x \le 1$, $0 \le y \le 1$ onto a region of the uv-plane. Express the integral

$$\iint\limits_{R_{xy}} \sqrt[3]{x^4 - 6x^2y^2 + y^4}\, dx\, dy$$

over the square as an iterated integral in u and v.

7. Transform the integrals given, using the substitutions indicated:

a) $\int_0^1 \int_0^x \log(1 + x^2 + y^2)\, dy\, dx$, $x = u + v$, $y = u - v$

b) $\int_0^1 \int_{1-x}^{1+x} \sqrt{1 + x^2y^2}\, dy\, dx$, $x = u$, $y = u + v$

8. Let R_{uv} be the square $0 \le u \le 1$, $0 \le v \le 1$. Show that the given equations define a one-to-one mapping of R_{uv} onto R_{xy} and graph R_{xy}:

a) $x = u + u^2$, $y = e^v$ **b)** $x = ue^v$, $y = e^v$

c) $x = 2u - v^2$, $y = v + uv$ **d)** $x = 5u - u^2 + v^2$, $y = 5v + 10uv$

9. Verify the correctness of (4.67) as a special case of (4.66). Show the geometric meaning of the volume element $r\,\Delta r\,\Delta\theta\,\Delta z$.

10. Verify the correctness of (4.68) as a special case of (4.66). Show the geometric meaning of the volume element $\rho^2 \sin\phi \Delta\rho\Delta\phi\Delta\theta$.

11. Transform to cylindrical coordinates but do not evaluate:

a) $\iiint_{R_{xyz}} x^2y\, dx\, dy\, dz$, where R_{xyz} is the region $x^2 + y^2 \le 1$, $0 \le z \le 1$;

b) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1+x+y} (x^2 - y^2)\, dz\, dy\, dx$.

12. Transform to spherical coordinates but do not evaluate:

a) $\iiint_{R_{xyz}} x^2y\, dx\, dy\, dz$, where R_{xyz} is the sphere: $x^2 + y^2 + z^2 \le a^2$;

b) $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$.

4.7 ARC LENGTH AND SURFACE AREA

In elementary calculus it is shown that a curve $y = f(x)$, $a \le x \le b$, has length

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4.69}$$

and that if the curve is given parametrically by equations $x = x(t)$, $y = y(t)$ for $t_1 \le t \le t_2$, then it has length

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{4.70}$$