$a \le t \le b$. We then define:

$$\int_a^b \mathbf{F}(t)\,dt = \int_a^b f(t)\,dt\,\mathbf{i} + \int_a^b g(t)\,dt\,\mathbf{j} + \int_a^b h(t)\,dt\,\mathbf{k}. \tag{4.57}$$

This integral can be interpreted as the limit of a sum:

$$\lim_{h\to 0} \sum_{i=1}^{n} \Delta_i t \mathbf{F}(t_i^*)$$

as in (4.1). The limit is a *vector* $\mathbf{c}$ and existence of the limit means that given $\epsilon > 0$, there is a $\delta > 0$ such that, for $0 < h < \delta$,

$$\left| \sum_{i=1}^{n} \Delta_i t \mathbf{F}(t_i^*) - \mathbf{c} \right| < \epsilon.$$

The properties (4.3), ..., (4.6), (4.10), (4.16) all extend at once to the integral of $\mathbf{F}(t)$, where the absolute value $|f(x)|$ is replaced by the vector norm $|\mathbf{F}(t)|$.

One can also integrate vector functions of several variables. For example, if

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

is continuous on the bounded closed region R, suitable for triple integrals, then

$$\iiint_R \mathbf{F}(x, y, z)\,dV = \iiint_R f(x, y, z)\,dV\mathbf{i} + \iiint_R g(x, y, z)\,dV\mathbf{j} + \iiint_R h(x, y, z)\,dV\mathbf{k}. \tag{4.58}$$

This integral can also be interpreted as the limit of a sum, and the familiar properties of triple integrals carry over.

PROBLEMS

1. Evaluate the following integrals:
   a) $\iint_R (x^2 + y^2)\,dx\,dy$, where R is the triangle with vertices (0, 0), (1, 0), (1, 1);
   b) $\iiint_R u^2 v^2 w\,du\,dv\,dw$, where R is the region: $u^2 + v^2 \le 1, 0 \le w \le 1$;
   c) $\iint_R r^3 \cos\theta\,dr\,d\theta$, where R is the region: $1 \le r \le 2, \frac{\pi}{4} \le \theta \le \pi$;
   d) $\iiint_R (x + z)\,dV$, where R is the tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 2, 0), (0, 0, 3).
2. Find the volume of the solid region below the given surface $z = f(x, y)$ for (x, y) in the region R defined by the given inequalities:
   a) $z = e^x \cos y,\ 0 \le x \le 1,\ 0 \le y \le \pi/2$
   b) $z = x^2 e^{-x-y},\ 0 \le x \le 1,\ 0 \le y \le 2$

c) $z = x^2 y,\ 0 \le x \le 1,\ x+1 \le y \le x+2$

d) $z = \sqrt{x^2 - y^2},\ x^2 - y^2 \ge 0,\ 0 \le x \le 1$

3. For each of the following choice of R, represent $\iint f(x,y)\,dA$ over R as an iterated integral of both forms (4.33) and (4.34):

a) $1 \le x \le 2,\ 1 - x \le y \le 1 + x$

b) $y^2 + x(x-1) \le 0$

4. Evaluate $\iiint_R f(x,y,z)\,dV$ for the following choices of f and R:

a) $f(x,y,z) = \sqrt{x+y+z}$, R the cube of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$.

b) $f(x,y,z) = x^2 + z^2$, R the pyramid of vertices $(\pm 1, \pm 1, 0)$ and $(0,0,1)$.

5. For each of the following iterated integrals, find the region R and write the integral in the other form (interchanging the order of integration):

a) $\int_{1/2}^{1} \int_0^{1-x} f(x,y)\,dy\,dx$

b) $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$

c) $\int_0^1 \int_{y-1}^{0} f(x,y)\,dx\,dy$

d) $\int_0^1 \int_{1-x}^{1+x} f(x,y)\,dy\,dx$

6. Express the following in terms of multiple integrals and reduce to iterated integrals, but do not evaluate:

a) the mass of a sphere whose density is proportional to the distance from one diametral plane;

b) the coordinates of the center of mass of the sphere of part (a);

c) the moment of inertia about the x-axis of the solid filling the region $0 \le z \le 1 - x^2 - y^2, 0 \le x \le 1, 0 \le y \le 1 - x$ and having density proportional to xy.

7. The moment of inertia of a solid about an arbitrary line L is defined as

$$I_L = \iiint_R d^2 f(x,y,z)\,dx\,dy\,dz,$$

where f is density and d is the distance from a general point (x,y,z) of the solid to the line L. Prove the *Parallel Axis theorem*:

$$I_L = I_{\overline{L}} + Mh^2,$$

where $\overline{L}$ is a line parallel to L through the center of mass, M is the mass, and h is the distance between L and $\overline{L}$. (Hint: Take $\overline{L}$ to be the z axis.)

8. Let L be a line through the origin O with direction cosines l, m, n. Prove that

$$I_L = I_x l^2 + I_y m^2 + I_z n^2 - 2I_{xy} lm - 2I_{yz} mn - 2I_{zx} ln,$$

where

$$I_{xy} = \iiint_R xyf(x,y,z)\,dx\,dy\,dz, \qquad I_{yz} = \iiint_R yzf \ldots .$$

The new integrals are called *products of inertia*. The locus

$$I_x x^2 + I_y y^2 + I_z z^2 - 2(I_{xy} xy + I_{yz} yz + I_{zx} zx) = 1$$

is an ellipsoid called the *ellipsoid of inertia*.

9. It is shown in geometry that the medians of a triangle meet at a point, which is the centroid of the triangle, and that the lines from the vertices of a tetrahedron to the centroids of the opposite faces meet at a point which is 3/4 of the way from each vertex to the opposite face along the lines described. Show that this last point is the centroid of the tetrahedron. [Hint: Take the base of the tetrahedron to be in the xy-plane and show that $\bar{z} = h/4$, if h is the z-coordinate of the vertex not in the xy-plane.]

10. Evaluate the integrals:

a) $\int_0^1 \mathbf{F}(t)\,dt$, if $\mathbf{F}(t) = t^2\mathbf{i} - e^t\mathbf{j} + \frac{1}{1+t}\mathbf{k}$.

b) $\iint_R \mathbf{F}(x, y)\,dA$, if R is the triangular region enclosed by the triangle of vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ and $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^2\mathbf{j}$.

11. Let $\mathbf{F}(t)$ be continuous for $a \le t \le b$ and let $\mathbf{q}$ be a constant vector. Prove:

a) $\int_a^b \mathbf{q} \cdot \mathbf{F}(t)\,dt = \mathbf{q} \cdot \int_a^b \mathbf{F}(t)\,dt$

b) $\int_a^b \mathbf{q} \times \mathbf{F}(t)\,dt = \mathbf{q} \times \int_a^b \mathbf{F}(t)\,dt$

4.6 Change of Variables in Integrals

For functions of one variable the chain rule

$$\frac{dF}{du} = \frac{dF}{dx}\frac{dx}{du} \tag{4.59}$$

at once gives the rule for change of variable in a definite integral:

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2} f[x(u)]\frac{dx}{du}\,du. \tag{4.60}$$

Here $f(x)$ is assumed to be continuous at least for $x_1 \le x \le x_2$, $x = x(u)$ is defined for $u_1 \le u \le u_2$ and has a continuous derivative, with $x_1 = x(u_1)$, $x_2 = x(u_2)$, and $f[x(u)]$ is continuous for $u_1 \le u \le u_2$.

Proof. If $F(x)$ is an indefinite integral of $f(x)$, then

$$\int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1).$$

But $F[x(u)]$ is then an indefinite integral of $f[x(u)]\frac{dx}{du}$, for (4.59) gives

$$\frac{dF}{du} = \frac{dF}{dx}\frac{dx}{du} = f(x)\frac{dx}{du} = f\big[x(u)\big]\frac{dx}{du},$$

when x is expressed in terms of u. Thus the integral on the right of (4.60) is

$$F[x(u_2)] - F[x(u_1)] = F(x_2) - F(x_1).$$

Since this is the same as the value of the left-hand side of (4.60), the rule is established. ●

It is worth noting that the emphasis in (4.60) is on the function $x(u)$ rather than on its inverse $u = u(x)$. Such an inverse will exist only when x is a steadily increasing function of u or a steadily decreasing function of u. This is not required for (4.60). In fact, the function $x(u)$ can take on values outside the interval $x_1 \le x \le x_2$, as illustrated in Fig. 4.7. However, $f[x(u)]$ must remain continuous for $u_1 \le u \le u_2$.