PROBLEMS

1. Give several examples of functions of several variables occurring in geometry (area and volume formulas, law of cosines, and so on).

2. Represent the following functions by first sketching a surface, and second, drawing level curves:

   a) $z = 3 - x - 3y$ b) $z = x^2 + y^2 + 1$

   c) $z = \sin(x + y)$ d) $z = e^{xy}$

3. Analyze the following functions by describing their level surfaces in space:

   a) $u = x^2 + y^2 + z^2$ b) $u = x + y + z$

   c) $w = x^2 + y^2 - z$ d) $w = x^2 + y^2$

4. Determine the values of the following limits, wherever the limit exists:

   a) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 - y^2}{1 + x^2 + y^2}$ b) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{x^2 + y^2}$

   c) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{(1 + y^2)\sin x}{x}$ d) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{1 + x - y}{x^2 + y^2}$

5. Show that the following functions are discontinuous at (0, 0) and graph the corresponding surfaces:[1]

   a) $z = \frac{x}{x - y}$ b) $z = \log(x^2 + y^2)$

6. Describe the sets in which the following functions are defined:

   a) $z = e^{x-y}$ b) $z = \log(x^2 + y^2 - 1)$

   c) $z = \sqrt{1 - x^2 - y^2}$ d) $u = \frac{xy}{z}$

7. Prove the theorem: Let $f(x, y)$ be defined in domain D and continuous at the point (x, y_1) of D. If $f(x_1, y_1) > 0$, then there is a neighborhood of (x_1, y_1) in which $f(x, y) > \frac{1}{2} f(x_1, y_1) > 0$. [Hint: Use $\epsilon = \frac{1}{2} f(x_1, y_1)$ in the definition of continuity.]

8. Let D be a domain in the plane. Show that D cannot consist of two open sets E_1, E_2 with no point in common. [Hint: Suppose the contrary and choose point P in E_1 and point Q in E_2; join these points by a broken line in D. Regard this line as a path from P to Q and let s be distance from P along the path, so that the path is given by continuous functions $x = x(s)$, $y = y(s)$, $0 \leq s \leq L$, with $s = 0$ at P and $s = L$ at Q. Let $f(s) = -1$ if $(x(s), y(s))$ is in E_1 and let $f(s) = 1$ if $(x(s), y(s))$ is in E_2. Show that $f(s)$ is continuous for $0 \leq s \leq L$. Now apply the *intermediate value theorem*: If $f(x)$ is continuous for $a \leq x \leq b$ and $f(a) < 0$, $f(b) > 0$, then $f(x) = 0$ for some x between a and b (see Problem 5 following Section 2.23).]

9. Prove the theorem: Let $f(x, y)$ be continuous in domain D. Let $f(x, y)$ be positive for at least one point of D and negative for at least one point of D. Then $f(x, y) = 0$ for at least one point of D. [Hint: Use Problem 7 to conclude that the set A where $f(x, y) > 0$ and the set B where $f(x, y) < 0$ are open. If $f(x, y) \neq 0$ in D, then D is formed of the two nonoverlapping open sets A and B; this is not possible by Problem 8.]

Remark This result extends the intermediate value theorem to functions of two variables.

---

[1] In this book, $\log x$ denotes the natural logarithm of x.