Exercises

A

For Exercises 1-4, use Green's Theorem to evaluate the given line integral around the curve C, traversed counterclockwise.

1. $\oint_C (x^2 - y^2)\,dx + 2xy\,dy$; C is the boundary of $R = \{(x,y) : 0 \le x \le 1,\ 2x^2 \le y \le 2x\}$

2. $\oint_C x^2 y\,dx + 2xy\,dy$; C is the boundary of $R = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$

3. $\oint_C 2y\,dx - 3x\,dy$; C is the circle $x^2 + y^2 = 1$

4. $\oint_C (e^{x^2} + y^2)\,dx + (e^{y^2} + x^2)\,dy$; C is the boundary of the triangle with vertices $(0,0)$, $(4,0)$ and $(0,4)$

5. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = (y^2 + 3x^2)\,\mathbf{i} + 2xy\,\mathbf{j}$? If so, find one.

6. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = (x^3\cos(xy) + 2x\sin(xy))\,\mathbf{i} + x^2 y\cos(xy)\,\mathbf{j}$? If so, find one.

7. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = (8xy + 3)\,\mathbf{i} + 4(x^2 + y)\,\mathbf{j}$? If so, find one.

8. Show that for any constants a, b and any closed simple curve C, $\oint_C a\,dx + b\,dy = 0$.

B

9. For the vector field $\mathbf{f}$ as in Example 4.8, show directly that $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$, where C is the boundary of the annulus $R = \{(x,y) : 1/4 \le x^2 + y^2 \le 1\}$ traversed so that R is always on the left.

10. Evaluate $\oint_C e^x \sin y\,dx + (y^3 + e^x \cos y)\,dy$, where C is the boundary of the rectangle with vertices $(1,-1)$, $(1,1)$, $(-1,1)$ and $(-1,-1)$, traversed counterclockwise.

C

11. For a region R bounded by a simple closed curve C, show that the area A of R is

$$A = -\oint_C y\,dx = \oint_C x\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx\,,$$

where C is traversed so that R is always on the left. (*Hint: Use Green's Theorem and the fact that* $A = \iint_R 1\,dA$.)

Finally, by Stokes' Theorem, we know that if C is a simple closed curve in some solid region S in $\mathbb{R}^3$ and if $\mathbf{f}(x,y,z)$ is a smooth vector field such that curl $\mathbf{f} = \mathbf{0}$ in S, then

$$\oint_C \mathbf{f}\cdot d\mathbf{r} \;=\; \iint_\Sigma (\text{curl}\,\mathbf{f})\cdot\mathbf{n}\,d\sigma \;=\; \iint_\Sigma \mathbf{0}\cdot\mathbf{n}\,d\sigma \;=\; \iint_\Sigma 0\,d\sigma \;=\; 0\,,$$

where Σ is any orientable surface inside S whose boundary is C (such a surface is sometimes called a *capping surface* for C). So similar to the two-variable case, we have a three-dimensional version of a result from Section 4.3, for solid regions in $\mathbb{R}^3$ which are **simply connected** (i.e. regions having no holes):

The following statements are equivalent for a simply connected solid region S in $\mathbb{R}^3$:

(a) $\mathbf{f}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ has a smooth potential $F(x,y,z)$ in S

(b) $\displaystyle\int_C \mathbf{f}\cdot d\mathbf{r}$ is independent of the path for any curve C in S

(c) $\displaystyle\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ for every simple closed curve C in S

(d) $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ in S (i.e. curl $\mathbf{f} = \mathbf{0}$ in S)

Part (d) is also a way of saying that the differential form $P\,dx + Q\,dy + R\,dz$ is exact.

Example 4.16. Determine if the vector field $\mathbf{f}(x,y,z) = xyz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ has a potential in $\mathbb{R}^3$.

Solution: Since $\mathbb{R}^3$ is simply connected, we just need to check whether curl $\mathbf{f} = \mathbf{0}$ throughout $\mathbb{R}^3$, that is,

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}\,, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}\,, \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

throughout $\mathbb{R}^3$, where $P(x,y,z) = xyz$, $Q(x,y,z) = xz$, and $R(x,y,z) = xy$. But we see that

$$\frac{\partial P}{\partial z} = xy\,,\; \frac{\partial R}{\partial x} = y \quad\Rightarrow\quad \frac{\partial P}{\partial z} \neq \frac{\partial R}{\partial x} \;\text{ for some } (x,y,z) \text{ in } \mathbb{R}^3.$$

Thus, $\mathbf{f}(x,y,z)$ does not have a potential in $\mathbb{R}^3$.

Exercises

A

For Exercises 1-3, calculate $\int_C f(x,y,z)\,ds$ for the given function $f(x,y,z)$ and curve C.

1. $f(x,y,z) = z$; $\quad C: x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le 2\pi$

2. $f(x,y,z) = \dfrac{x}{y} + y + 2yz;\quad C : x = t^2,\ y = t,\ z = 1,\ 1 \le t \le 2$

3. $f(x,y,z) = z^2;\quad C : x = t\sin t,\ y = t\cos t,\ z = \frac{2\sqrt{2}}{3}t^{3/2},\ 0 \le t \le 1$

For Exercises 4-9, calculate $\int_C \mathbf{f}\cdot d\mathbf{r}$ for the given vector field $\mathbf{f}(x,y,z)$ and curve C.

4. $\mathbf{f}(x,y,z) = \mathbf{i} - \mathbf{j} + \mathbf{k};\quad C : x = 3t,\ y = 2t,\ z = t,\ 0 \le t \le 1$

5. $\mathbf{f}(x,y,z) = y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k};\quad C : x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le 2\pi$

6. $\mathbf{f}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k};\quad C : x = \cos t,\ y = \sin t,\ z = 2,\ 0 \le t \le 2\pi$

7. $\mathbf{f}(x,y,z) = (y-2z)\mathbf{i} + xy\mathbf{j} + (2xz+y)\mathbf{k};\quad C : x = t,\ y = 2t,\ z = t^2 - 1,\ 0 \le t \le 1$

8. $\mathbf{f}(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k};$ C : the polygonal path from $(0,0,0)$ to $(1,0,0)$ to $(1,2,0)$

9. $\mathbf{f}(x,y,z) = xy\,\mathbf{i} + (z-x)\mathbf{j} + 2yz\,\mathbf{k};$ C : the polygonal path from $(0,0,0)$ to $(1,0,0)$ to $(1,2,0)$ to $(1,2,-2)$

For Exercises 10-13, state whether or not the vector field $\mathbf{f}(x,y,z)$ has a potential in $\mathbb{R}^3$ (you do not need to find the potential itself).

10. $\mathbf{f}(x,y,z) = y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}$

11. $\mathbf{f}(x,y,z) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ (a, b, c constant)

12. $\mathbf{f}(x,y,z) = (x+y)\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$

13. $\mathbf{f}(x,y,z) = xy\,\mathbf{i} - (x - yz^2)\mathbf{j} + y^2z\,\mathbf{k}$

B

For Exercises 14-15, verify Stokes' Theorem for the given vector field $\mathbf{f}(x,y,z)$ and surface Σ.

14. $\mathbf{f}(x,y,z) = 2y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k};\quad \Sigma : x^2 + y^2 + z^2 = 1,\ z \ge 0$

15. $\mathbf{f}(x,y,z) = xy\,\mathbf{i} + xz\,\mathbf{j} + yz\,\mathbf{k};\quad \Sigma : z = x^2 + y^2,\ z \le 1$

16. Construct a Möbius strip from a piece of paper, then draw a line down its center (like the dotted line in Figure 4.5.3(b)). Cut the Möbius strip along that center line completely around the strip. How many surfaces does this result in? How would you describe them? Are they orientable?

17. Use Gnuplot (see Appendix C) to plot the Möbius strip parametrized as:

$$\mathbf{r}(u,v) \;=\; \cos u\,(1 + v\cos\tfrac{u}{2})\mathbf{i} + \sin u\,(1 + v\cos\tfrac{u}{2})\mathbf{j} + v\sin\tfrac{u}{2}\,\mathbf{k}\ ,\quad 0 \le u \le 2\pi\ ,\quad -\tfrac{1}{2} \le v \le \tfrac{1}{2}$$

C

18. Let Σ be a closed surface and $\mathbf{f}(x,y,z)$ a smooth vector field. Show that $\iint\limits_{\Sigma} (\text{curl } \mathbf{f})\cdot\mathbf{n}\,d\sigma = 0$. (*Hint: Split Σ in half.*)

19. Show that Green's Theorem is a special case of Stokes' Theorem.

for any constant K, so $F(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a potential for $\mathbf{f}(x,y)$. Thus,

$$\oint_C x\,dx + y\,dy = \oint_C \mathbf{f}\cdot d\mathbf{r} = 0$$

by Corollary 4.6, since the curve C is closed (it is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$).

Exercises

A

1. Evaluate $\oint_C (x^2 + y^2)\,dx + 2xy\,dy$ for $C: x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

2. Evaluate $\int_C (x^2 + y^2)\,dx + 2xy\,dy$ for $C: x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$.

3. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = y\,\mathbf{i} - x\,\mathbf{j}$? If so, find one.

4. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = x\,\mathbf{i} - y\,\mathbf{j}$? If so, find one.

5. Is there a potential $F(x,y)$ for $\mathbf{f}(x,y) = xy^2\,\mathbf{i} + x^3y\,\mathbf{j}$? If so, find one.

B

6. Let $\mathbf{f}(x,y)$ and $\mathbf{g}(x,y)$ be vector fields, let a and b be constants, and let C be a curve in $\mathbb{R}^2$. Show that

$$\int_C (a\,\mathbf{f} \pm b\,\mathbf{g})\cdot d\mathbf{r} = a\int_C \mathbf{f}\cdot d\mathbf{r} \pm b\int_C \mathbf{g}\cdot d\mathbf{r}\,.$$

7. Let C be a curve whose arc length is L. Show that $\int_C 1\,ds = L$.

8. Let $f(x,y)$ and $g(x,y)$ be continuously differentiable real-valued functions in a region R. Show that

$$\oint_C f\,\nabla g\cdot d\mathbf{r} = -\oint_C g\,\nabla f\cdot d\mathbf{r}$$

for any closed curve C in R. (*Hint: Use Exercise 21 in Section 2.4.*)

9. Let $\mathbf{f}(x,y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ for all $(x,y) \ne (0,0)$, and $C: x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

(a) Show that $\mathbf{f} = \nabla F$, for $F(x,y) = \tan^{-1}(y/x)$.

(b) Show that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 2\pi$. Does this contradict Corollary 4.6? Explain.

C

10. Let $g(x)$ and $h(y)$ be differentiable functions, and let $\mathbf{f}(x,y) = h(y)\,\mathbf{i} + g(x)\,\mathbf{j}$. Can $\mathbf{f}$ have a potential $F(x,y)$? If so, find it. You may assume that F would be smooth. (*Hint: Consider the mixed partial derivatives of F.*)

where V is the volume enclosed by a closed surface Σ around the point (x, y, z). In the limit, $V \to 0$ means that we take smaller and smaller closed surfaces around (x, y, z), which means that the volumes they enclose are going to zero. It can be shown that this limit is independent of the shapes of those surfaces. Notice that the limit being taken is of the ratio of the flux through a surface to the volume enclosed by that surface, which gives a rough measure of the flow "leaving" a point, as we mentioned. Vector fields which have zero divergence are often called *solenoidal* fields.

The following theorem is a simple consequence of formula (4.33).

Theorem 4.9. If the flux of a vector field $\mathbf{f}$ is zero through every closed surface containing a given point, then div $\mathbf{f} = 0$ at that point.

Proof: By formula (4.33), at the given point (x, y, z) we have

$$\begin{aligned}
\text{div}\,\mathbf{f}(x,y,z) &= \lim_{V\to 0} \frac{1}{V} \iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma} \quad \text{for closed surfaces } \Sigma \text{ containing } (x,y,z), \text{ so} \\
&= \lim_{V\to 0} \frac{1}{V}\,(0) \quad \text{by our assumption that the flux through each } \Sigma \text{ is zero, so} \\
&= \lim_{V\to 0} 0 \\
&= 0\,.
\end{aligned}$$

QED

Lastly, we note that sometimes the notation

$$\oiint_{\Sigma} f(x,y,z)\,d\sigma \quad \text{and} \quad \oiint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$$

is used to denote surface integrals of scalar and vector fields, respectively, over closed surfaces. Especially in physics texts, it is common to see simply $\oint_{\Sigma}$ instead of $\oiint_{\Sigma}$.

Exercises

A

For Exercises 1-4, use the Divergence Theorem to evaluate the surface integral $\iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$ of the given vector field $\mathbf{f}(x, y, z)$ over the surface Σ.

1. $\mathbf{f}(x, y, z) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 9$

2. $\mathbf{f}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, Σ : boundary of the solid cube $S = \{(x, y, z) : 0 \le x, y, z \le 1\}$

3. $\mathbf{f}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 1$

4. $\mathbf{f}(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 1$

B

5. Show that the flux of any constant vector field through any closed surface is zero.

6. Evaluate the surface integral from Exercise 2 *without* using the Divergence Theorem, i.e. using only Definition 4.3, as in Example 4.10. Note that there will be a different outward unit normal vector to each of the six faces of the cube.

7. Evaluate the surface integral $\iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$, where $\mathbf{f}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and Σ is the part of the plane $6x + 3y + 2z = 6$ with $x \ge 0$, $y \ge 0$, and $z \ge 0$, with the outward unit normal $\mathbf{n}$ pointing in the positive z direction.

8. Use a surface integral to show that the surface area of a sphere of radius r is $4\pi r^2$. (*Hint: Use spherical coordinates to parametrize the sphere.*)

9. Use a surface integral to show that the surface area of a right circular cone of radius R and height h is $\pi R\sqrt{h^2 + R^2}$. (*Hint: Use the parametrization* $x = r\cos\theta$, $y = r\sin\theta$, $z = \frac{h}{R}r$, *for* $0 \le r \le R$ *and* $0 \le \theta \le 2\pi$.)

10. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ can be parametrized using *ellipsoidal coordinates*

$$x = a\sin\phi\cos\theta\ ,\ y = b\sin\phi\sin\theta\ ,\ z = c\cos\phi\ ,\ \text{for } 0 \le \theta \le 2\pi \text{ and } 0 \le \phi \le \pi.$$

Show that the surface area S of the ellipsoid is

$$S = \int_0^{\pi}\int_0^{2\pi} \sin\phi\sqrt{a^2b^2\cos^2\phi + c^2(a^2\sin^2\theta + b^2\cos^2\theta)\sin^2\phi}\, d\theta\, d\phi\ .$$

(Note: The above double integral can not be evaluated by elementary means. For specific values of a, b and c it can be evaluated using numerical methods. An alternative is to express the surface area in terms of *elliptic integrals*.[5])

C

11. Use Definition 4.3 to prove that the surface area S over a region R in $\mathbb{R}^2$ of a surface $z = f(x, y)$ is given by the formula

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA\ .$$

(*Hint: Think of the parametrization of the surface.*)

[5] BOWMAN, F., *Introduction to Elliptic Functions, with Applications*, New York: Dover, 1961, § III.7.