Superposition Principle. If y_1 and y_2 are solutions to the equations

$$ay'' + by' + cy = g_1 \quad \text{and} \quad ay'' + by' + cy = g_2 ,$$

respectively, then $c_1y_1 + c_2y_2$ is a solution to the equation

$$ay'' + by' + cy = c_1g_1 + c_2g_2 .$$

The superposition principle facilitates finding a particular solution when the nonhomogeneous term is the sum of nonhomogeneities for which particular solutions can be determined.

Cauchy–Euler (Equidimensional) Equations

$$at^2y'' + bty' + cy = g(t)$$

Substituting $y = t^r$ yields the associated characteristic equation

$$ar^2 + (b - a)r + c = 0$$

for the corresponding *homogeneous* Cauchy–Euler equation. A general solution to the homogeneous equation for $t > 0$ is given by

(i) $c_1t^{r_1} + c_2t^{r_2}$, if r_1 and r_2 are distinct real roots;
(ii) $c_1t^r + c_2t^r \ln t$, if r is a repeated root;
(iii) $c_1t^\alpha \cos(\beta \ln t) + c_2t^\alpha \sin(\beta \ln t)$, if $\alpha + i\beta$ is a complex root.

A general solution to the nonhomogeneous equation is $y = y_p + y_h$, where y_p is a particular solution and y_h is a general solution to the corresponding homogeneous equation. The method of variation of parameters (but not the method of undetermined coefficients) can be used to find a particular solution.

REVIEW PROBLEMS

In Problems 1–28, find a general solution to the given differential equation.

1. $y'' + 8y' - 9y = 0$
2. $49y'' + 14y' + y = 0$
3. $4y'' - 4y' + 10y = 0$
4. $9y'' - 30y' + 25y = 0$
5. $6y'' - 11y' + 3y = 0$
6. $y'' + 8y' - 14y = 0$
7. $36y'' + 24y' + 5y = 0$
8. $25y'' + 20y' + 4y = 0$
9. $16z'' - 56z' + 49z = 0$
10. $u'' + 11u = 0$
11. $t^2x''(t) + 5x(t) = 0 , \quad t > 0$
12. $2y''' - 3y'' - 12y' + 20y = 0$
13. $y'' + 16y = te^t$
14. $v'' - 4v' + 7v = 0$
15. $3y''' + 10y'' + 9y' + 2y = 0$
16. $y''' + 3y'' + 5y' + 3y = 0$
17. $y''' + 10y' - 11y = 0$
18. $y^{(4)} = 120t$
19. $4y''' + 8y'' - 11y' + 3y = 0$
20. $2y'' - y = t \sin t$
21. $y'' - 3y' + 7y = 7t^2 - e^t$
22. $y'' - 8y' - 33y = 546 \sin t$
23. $y''(\theta) + 16y(\theta) = \tan 4\theta$
24. $10y'' + y' - 3y = t - e^{t/2}$
25. $4y'' - 12y' + 9y = e^{5t} + e^{3t}$

26. $y'' + 6y' + 15y = e^{2t} + 75$

27. $x^2y'' + 2xy' - 2y = 6x^{-2} + 3x\ , \qquad x > 0$

28. $y'' = 5x^{-1}y' - 13x^{-2}y\ , \qquad x > 0$

In Problems 29–36, find the solution to the given initial value problem.

29. $y'' + 4y' + 7y = 0\ ;$
$y(0) = 1\ , \qquad y'(0) = -2$

30. $y''(\theta) + 2y'(\theta) + y(\theta) = 2\cos\theta\ ;$
$y(0) = 3\ , \qquad y'(0) = 0$

31. $y'' - 2y' + 10y = 6\cos 3t - \sin 3t\ ;$
$y(0) = 2\ , \qquad y'(0) = -8$

32. $4y'' - 4y' + 5y = 0\ ;$
$y(0) = 1\ , \qquad y'(0) = -11/2$

33. $y''' - 12y'' + 27y' + 40y = 0\ ;$
$y(0) = -3\ , \qquad y'(0) = -6\ , \qquad y''(0) = -12$

34. $y'' + 5y' - 14y = 0\ ;$
$y(0) = 5\ , \qquad y'(0) = 1$

35. $y''(\theta) + y(\theta) = \sec\theta\ ; \quad y(0) = 1\ , \quad y'(0) = 2$

36. $9y'' + 12y' + 4y = 0\ ;$
$y(0) = -3\ , \qquad y'(0) = 3$

37. Use the mass–spring oscillator analogy to decide whether all solutions to each of the following differential equations are bounded as $t \to +\infty$.

(a) $y'' + t^4y = 0$ **(b)** $y'' - t^4y = 0$

(c) $y'' + y^7 = 0$ **(d)** $y'' + y^8 = 0$

(e) $y'' + (3 + \sin t)y = 0$ **(f)** $y'' + t^2y' + y = 0$

(g) $y'' - t^2y' - y = 0$

38. A 3-kg mass is attached to a spring with stiffness $k = 75$ N/m, as in Figure 4.1, page 163. The mass is displaced 1/4 m to the left and given a velocity of 1 m/sec to the right. The damping force is negligible. Find the equation of motion of the mass along with the amplitude, period, and frequency. How long after release does the mass pass through the equilibrium position?

39. A 32-lb weight is attached to a vertical spring, causing it to stretch 6 in. upon coming to rest at equilibrium. The damping constant for the system is 2 lb-sec/ft. An external force $F(t) = 4\cos 8t$ lb is applied to the weight. Find the steady-state solution for the system. What is its resonant frequency?

TECHNICAL WRITING EXERCISES

1. Compare the two methods—undetermined coefficients and variation of parameters—for determining a particular solution to a nonhomogeneous equation. What are the advantages and disadvantages of each?

2. Consider the differential equation

$$\frac{d^2y}{dx^2} + 2b\frac{dy}{dx} + y = 0\ ,$$

where b is a constant. Describe how the behavior of solutions to this equation changes as b varies.

3. Consider the differential equation

$$\frac{d^2y}{dx^2} + cy = 0\ ,$$

where c is a constant. Describe how the behavior of solutions to this equation changes as c varies.

4. For students with a background in linear algebra: Compare the theory for linear second-order equations with that for systems of n linear equations in n unknowns whose coefficient matrix has rank $n - 2$. Use the terminology from linear algebra; for example, subspace, basis, dimension, linear transformation, and kernel. Discuss both homogeneous and nonhomogeneous equations.