

Thus, the solution of the Fokker–Planck equation is given by

$$u(x, t) = \sum_{n=1}^{\infty} a_n \exp\left(-nt - \frac{1}{2}x^2\right) H_n\left(\frac{x}{\sqrt{2}}\right), \quad (7.8.29)$$

where H_n is the Hermite function and a_n are arbitrary constants to be determined from the given initial condition

$$u(x, 0) = f(x). \quad (7.8.30)$$

We make the change of variables

$$\xi = x e^t \quad \text{and} \quad u = e^t v, \quad (7.8.31)$$

in equation (7.8.25). Consequently, equation (7.8.25) becomes

$$\frac{\partial v}{\partial t} = e^{2t} \frac{\partial^2 v}{\partial \xi^2}. \quad (7.8.32)$$

Making another change of variable t to $\tau(t)$, we transform (7.8.32) into the linear diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2}. \quad (7.8.33)$$

Finally, we note that the asymptotic behavior of the solution $u(x, t)$ as $t \rightarrow \infty$ is of special interest. The reader is referred to Reif (1965) for such behavior.

7.9 Exercises

1. Solve the following initial boundary-value problems:

- (a) $u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0,$
 $u(x, 0) = x(1 - x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq 1,$
 $u(0, t) = u(1, t) = 0, \quad t > 0.$
- (b) $u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
 $u(x, 0) = 3 \sin x, \quad u_t(x, 0) = 0, \quad 0 \leq x \leq \pi,$
 $u(0, t) = u(\pi, t) = 0, \quad t > 0.$

2. Determine the solutions of the following initial boundary-value problems:

$$\begin{aligned}
 \text{(a)} \quad & u_{tt} = c^2 u_{xx}, & 0 < x < \pi, & & t > 0, \\
 & u(x, 0) = 0, & u_t(x, 0) = 8 \sin^2 x, & & 0 \leq x \leq \pi, \\
 & u(0, t) = u(\pi, t) = 0, & & & t > 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & u_{tt} = c^2 u_{xx} = 0, & 0 < x < 1, & & t > 0, \\
 & u(x, 0) = 0, & u_t(x, 0) = x \sin \pi x, & & 0 \leq x \leq 1, \\
 & u(0, t) = u(1, t) = 0, & & & t > 0.
 \end{aligned}$$

3. Find the solution of each of the following problems:

$$\begin{aligned}
 \text{(a)} \quad & u_{tt} = c^2 u_{xx} = 0, & 0 < x < 1, & & t > 0, \\
 & u(x, 0) = x(1-x), & u_t(x, 0) = x - \tan \frac{\pi x}{4}, & & 0 \leq x \leq 1, \\
 & u(0, t) = u(\pi, t) = 0, & & & t > 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & u_{tt} = c^2 u_{xx} = 0, & 0 < x < \pi, & & t > 0, \\
 & u(x, 0) = \sin x, & u_t(x, 0) = x^2 - \pi x, & & 0 \leq x \leq \pi, \\
 & u(0, t) = u(\pi, t) = 0, & & & t > 0.
 \end{aligned}$$

4. Solve the following problems:

$$\begin{aligned}
 \text{(a)} \quad & u_{tt} = c^2 u_{xx} = 0, & 0 < x < \pi, & & t > 0, \\
 & u(x, 0) = x + \sin x, & u_t(x, 0) = 0, & & 0 \leq x \leq \pi, \\
 & u(0, t) = u_x(\pi, t) = 0, & & & t > 0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & u_{tt} = c^2 u_{xx} = 0, & 0 < x < \pi, & & t > 0, \\
 & u(x, 0) = \cos x, & u_t(x, 0) = 0, & & 0 \leq x \leq \pi, \\
 & u_x(0, t) = 0, & u_x(\pi, t) = 0, & & t > 0.
 \end{aligned}$$

5. By the method of separation of variables, solve the telegraph equation:

$$\begin{aligned} u_{tt} + au_t + bu &= c^2 u_{xx}, & 0 < x < l, & \quad t > 0, \\ u(x, 0) &= f(x), & u_t(x, 0) &= 0, \\ u(0, t) &= u(l, t) = 0, & & \quad t > 0. \end{aligned}$$

6. Obtain the solution of the damped wave motion problem:

$$\begin{aligned} u_{tt} + au_t &= c^2 u_{xx}, & 0 < x < l, & \quad t > 0, \\ u(x, 0) &= 0, & u_t(x, 0) &= g(x), \\ u(0, t) &= u(l, t) = 0. \end{aligned}$$

7. The torsional oscillation of a shaft of circular cross section is governed by the partial differential equation

$$\theta_{tt} = a^2 \theta_{xx},$$

where $\theta(x, t)$ is the angular displacement of the cross section and a is a physical constant. The ends of the shaft are fixed elastically, that is,

$$\theta_x(0, t) - h\theta(0, t) = 0, \quad \theta_x(l, t) + h\theta(l, t) = 0.$$

Determine the angular displacement if the initial angular displacement is $f(x)$.

8. Solve the initial boundary-value problem of the longitudinal vibration of a truncated cone of length l and base of radius a . The equation of motion is given by

$$\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial}{\partial x} \left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x} \right], \quad 0 < x < l, \quad t > 0,$$

where $c^2 = (E/\rho)$, E is the elastic modulus, ρ is the density of the material and $h = la/(a-l)$. The two ends are rigidly fixed. If the initial displacement is $f(x)$, that is, $u(x, 0) = f(x)$, find $u(x, t)$.

9. Establish the validity of the formal solution of the initial boundary-value problems:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < \pi, & \quad t > 0, \\ u(x, 0) &= f(x), & u_t(x, 0) &= g(x), & 0 \leq x \leq \pi, \\ u_x(0, t) &= 0, & u_x(\pi, t) &= 0, & \quad t > 0. \end{aligned}$$

10. Prove the uniqueness of the solution of the initial boundary-value problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < \pi, & \quad t > 0, \\ u(x, 0) &= f(x), & u_t(x, 0) &= g(x), & 0 \leq x \leq \pi, \\ u_x(0, t) &= 0, & u_x(\pi, t) &= 0, & \quad t > 0. \end{aligned}$$

11. Determine the solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + A \sinh x, & 0 < x < l, & \quad t > 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, & 0 \leq x \leq l, & \\ u(0, t) &= h, \quad u(l, t) = k, & & \quad t > 0, \end{aligned}$$

where h , k , and A are constants.

12. Solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + Ax, & 0 < x < 1, \quad t > 0, \quad A = \text{constant}, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0, & 0 \leq x \leq 1, \\ u(0, t) &= 0, \quad u(1, t) = 0, & t > 0. \end{aligned}$$

13. Solve the problem:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + x^2, & 0 < x < 1, \quad t > 0, \\ u(x, 0) &= x, \quad u_t(x, 0) = 0, & 0 \leq x \leq 1, \\ u(0, t) &= 0, \quad u(1, t) = 1, & t \geq 0. \end{aligned}$$

14. Find the solution of the following problems:

$$\begin{aligned} \text{(a)} \quad u_t &= k u_{xx} + h, & 0 < x < 1, \quad t > 0, \quad h = \text{constant}, \\ u(x, 0) &= u_0 (1 - \cos \pi x), & 0 \leq x \leq 1, & \quad u_0 = \text{constant}, \\ u(0, t) &= 0, & u(1, t) = 2u_0, & \quad t \geq 0. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad u_t &= k u_{xx} - hu, & 0 < x < l, \quad t > 0, \quad h = \text{constant}, \\ u(x, 0) &= f(x), & 0 \leq x \leq l, \\ u_x(0, t) &= u_x(l, t) = 0, & t > 0. \end{aligned}$$

15. Obtain the solution of each of the following initial boundary-value problems:

$$\begin{aligned} \text{(a)} \quad u_t &= 4 u_{xx}, & 0 < x < 1, \quad t > 0, \\ u(x, 0) &= x^2 (1 - x), & 0 \leq x \leq 1, \\ u(0, t) &= 0, & u(1, t) = 0, & \quad t \geq 0. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad u_t &= k u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(x, 0) &= \sin^2 x, & 0 \leq x \leq \pi, \\ u(0, t) &= 0, & u(\pi, t) = 0, & \quad t \geq 0. \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & u_t = u_{xx}, & 0 < x < 2, & \quad t > 0, \\
 & u(x, 0) = x, & 0 \leq x \leq 2, & \\
 & u(0, t) = 0, & u_x(2, t) = 1, & \quad t \geq 0. \\
 \text{(d)} \quad & u_t = k u_{xx}, & 0 < x < l, & \quad t > 0, \\
 & u(x, 0) = \sin(\pi x/2l), & 0 \leq x \leq l, & \\
 & u(0, t) = 0, & u(l, t) = 1, & \quad t \geq 0.
 \end{aligned}$$

16. Find the temperature distribution in a rod of length l . The faces are insulated, and the initial temperature distribution is given by $x(l-x)$.
17. Find the temperature distribution in a rod of length π , one end of which is kept at zero temperature and the other end of which loses heat at a rate proportional to the temperature at that end $x = \pi$. The initial temperature distribution is given by $f(x) = x$.
18. The voltage distribution in an electric transmission line is given by

$$v_t = k v_{xx}, \quad 0 < x < l, \quad t > 0.$$

A voltage equal to zero is maintained at $x = l$, while at the end $x = 0$, the voltage varies according to the law

$$v(0, t) = Ct, \quad t > 0,$$

where C is a constant. Find $v(x, t)$ if the initial voltage distribution is zero.

19. Establish the validity of the formal solution of the initial boundary-value problem:

$$\begin{aligned}
 & u_t = k u_{xx}, & 0 < x < l, & \quad t > 0, \\
 & u(x, 0) = f(x), & 0 \leq x \leq l, & \\
 & u(0, t) = 0, & u_x(l, t) = 0, & \quad t \geq 0.
 \end{aligned}$$

20. Prove the uniqueness of the solution of the problem:

$$\begin{aligned}
 & u_t = k u_{xx}, & 0 < x < l, & \quad t > 0, \\
 & u(x, 0) = f(x), & 0 \leq x \leq l, & \\
 & u_x(0, t) = 0, & u_x(l, t) = 0, & \quad t \geq 0.
 \end{aligned}$$

21. Solve the radioactive decay problem:

$$\begin{aligned}u_t - k u_{xx} &= A e^{-ax}, & 0 < x < \pi, & & t > 0, \\u(x, 0) &= \sin x, & 0 \leq x \leq \pi, & & \\u(0, t) &= 0, & u(\pi, t) = 0, & & t \geq 0.\end{aligned}$$

22. Determine the solution of the initial boundary-value problem:

$$\begin{aligned}u_t - k u_{xx} &= h(x, t), & 0 < x < l, & & t > 0, & k = \text{constant}, \\u(x, 0) &= f(x), & 0 \leq x \leq l, & & \\u(0, t) &= p(t), & u(l, t) = q(t), & & t \geq 0.\end{aligned}$$

23. Determine the solution of the initial boundary-value problem:

$$\begin{aligned}u_t - k u_{xx} &= h(x, t), & 0 < x < l, & & t > 0, \\u(x, 0) &= f(x), & 0 \leq x \leq l, & & \\u(0, t) &= p(t), & u_x(l, t) = q(t), & & t \geq 0.\end{aligned}$$

24. Solve the problem:

$$\begin{aligned}u_t - k u_{xx} &= 0, & 0 < x < 1, & & t > 0, \\u(x, 0) &= x(1 - x), & 0 \leq x \leq 1, & & \\u(0, t) &= t, & u(1, t) = \sin t, & & t \geq 0.\end{aligned}$$

25. Solve the problem:

$$\begin{aligned}u_t - 4u_{xx} &= xt, & 0 < x < 1, & & t \geq 0, \\u(x, 0) &= \sin \pi x, & 0 \leq x \leq 1, & & \\u(0, t) &= t, & u(1, t) = t^2, & & t \geq 0.\end{aligned}$$

26. Solve the problem:

$$\begin{aligned}u_t - k u_{xx} &= x \cos t, & 0 < x < \pi, & & t > 0, \\u(x, 0) &= \sin x, & 0 \leq x \leq \pi, & & \\u(0, t) &= t^2, & u(\pi, t) = 2t, & & t \geq 0.\end{aligned}$$

27. Solve the problem:

$$\begin{aligned}u_t - u_{xx} &= 2x^2t, & 0 < x < 1, & & t > 0, \\u(x, 0) &= \cos(3\pi x/2), & 0 \leq x \leq 1, & & \\u(0, t) &= 1, & u_x(1, t) = \frac{3\pi}{2}, & & t \geq 0.\end{aligned}$$

28. Solve the problem:

$$\begin{aligned}u_t - 2u_{xx} &= h, & 0 < x < 1, & & t > 0, & h = \text{constant}, \\u(x, 0) &= x, & 0 \leq x \leq 1, & & \\u(0, t) &= \sin t, & u_x(1, t) + u(1, t) = 2, & & t \geq 0.\end{aligned}$$

29. Determine the solution of the initial boundary-value problem:

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= h(x, t), & 0 < x < l, & & t > 0, \\u(x, 0) &= f(x), & 0 \leq x \leq l, & & \\u_t(x, 0) &= g(x), & 0 \leq x \leq l, & & \\u(0, t) &= p(t), & u_x(l, t) &= q(t), & t \geq 0.\end{aligned}$$

30. Determine the solution of the initial boundary-value problem:

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= h(x, t), & 0 < x < l, & & t > 0, \\u(x, 0) &= f(x), & 0 \leq x \leq l, & & \\u_t(x, 0) &= g(x), & 0 \leq x \leq l, & & \\u_x(0, t) &= p(t), & u_x(l, t) &= q(t), & t \geq 0.\end{aligned}$$

31. Solve the problem:

$$\begin{aligned}u_{tt} - u_{xx} &= 0, & 0 < x < 1, & & t > 0, \\u(x, 0) &= x, & u_t(x, 0) &= 0, & 0 \leq x \leq 1, \\u(0, t) &= t^2, & u(1, t) &= \cos t, & t \geq 0.\end{aligned}$$

32. Solve the problem:

$$\begin{aligned}u_{tt} - 4u_{xx} &= xt, & 0 < x < 1, & & t > 0, \\u(x, 0) &= x, & u_t(x, 0) &= 0, & 0 \leq x \leq 1, \\u(0, t) &= 0, & u_x(1, t) &= 1 + t, & t \geq 0.\end{aligned}$$

33. Solve the problem:

$$\begin{aligned}u_{tt} - 9u_{xx} &= 0, & 0 < x < 1, & & t > 0, \\u(x, 0) &= \sin\left(\frac{\pi x}{2}\right), & u_t(x, 0) &= 1 + x, & 0 \leq x \leq 1, \\u_x(0, t) &= \pi/2, & u_x(1, t) &= 0, & t \geq 0.\end{aligned}$$

34. Find the solution of the problem:

$$\begin{aligned}u_{tt} + 2k u_t - c^2 u_{xx} &= 0, & 0 < x < l, & & t > 0, \\u(x, 0) &= 0, & u_t(x, 0) &= 0, & 0 \leq x \leq l, \\u_x(0, t) &= 0, & u(l, t) &= h, & t \geq 0, \quad h = \text{constant}.\end{aligned}$$

35. Solve the problem:

$$\begin{aligned}u_t - c^2 u_{xx} + hu &= hu_0, & -\pi < x < \pi, & & t > 0, \\u(x, 0) &= f(x), & -\pi \leq x \leq \pi, & & \\u(-\pi, t) &= u(\pi, t), & u_x(-\pi, t) &= u_x(\pi, t), & t \geq 0,\end{aligned}$$

where h and u_0 are constants.