When $\alpha = 1$, solution (12.17.87) becomes

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x-\xi,\, t)\, \psi_0(\xi)\, d\xi, \tag{12.17.89}$$

where the Green's function $G(x,t)$ is given by

$$\begin{aligned} G(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} E_{1,1}\left(-ak^2 t\right) dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(ikx - atk^2\right) dk \\ &= \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{x^2}{4at}\right). \end{aligned} \tag{12.17.90}$$

This solution (12.17.89) is in perfect agreement with the classical solution obtained by Debnath (1995).

12.18 Exercises

1. Find the Fourier transform of

   (a) $f(x) = \exp\left(-ax^2\right)$, (b) $f(x) = \exp(-a\,|x|)$,

   where a is a constant.
2. Find the Fourier transform of the gate function

$$f_a(x) = \begin{cases} 1, & |x| < a, \quad a \text{ is a positive constant.} \\ 0, & |x| \geq a. \end{cases}$$

3. Find the Fourier transform of

   (a) $f(x) = \frac{1}{|x|}$, (b) $f(x) = \chi_{[-a,a]}(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise,} \end{cases}$

   (c) $f(x) = \begin{cases} 1 - \frac{|x|}{a}, & |x| \leq a \\ 0, & |x| \geq a, \end{cases}$ (d) $f(x) = \frac{1}{(x^2+a^2)}$.

4. Find the Fourier transform of

   (a) $f(x) = \sin\left(x^2\right)$, (b) $f(x) = \cos\left(x^2\right)$.

5. Show that
$$I=\int_0^\infty e^{-a^2x^2}\,dx=\sqrt{\pi}/2a, \qquad a>0,$$
by noting that
$$I^2=\int_0^\infty\int_0^\infty e^{-a^2(x^2+y^2)}\,dx\,dy=\int_0^{\pi/2}\int_0^\infty e^{-a^2r^2}\,r\,dr\,d\theta.$$

6. Show that
$$\int_0^\infty e^{-a^2x^2}\cos bx\,dx=\left(\sqrt{\pi}/2a\right)e^{-b^2/4a^2}, \qquad a>0.$$

7. Prove that

(a) $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)\,dk=\mathcal{F}^{-1}\{F(k)\}$,

(b) $\mathcal{F}[f(ax-b)]=\frac{1}{|a|}e^{ikb/a}F(k/a)$.

8. Prove the following properties of the Fourier convolution:

(a) $f(x)*g(x)=g(x)*f(x)$, (b) $f*(g*h)=(f*g)*h$,

(c) $f*(ag+bh)=a(f*g)+b(f*h)$, where a and b are constants,

(d) $f*0=0*f=0$, (e) $f*1\neq f$,

(f) $f*\sqrt{2\pi}\,\delta=f=\sqrt{2\pi}\,\delta*f$,

(g) $\mathcal{F}\{f(x)\,g(x)\}=(F*G)(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k-\xi)\,G(\xi)\,d\xi$,

9. Prove the following properties of the Fourier convolution:

(a) $\frac{d}{dx}\{f(x)*g(x)\}=f'(x)*g(x)=f(x)*g'(x)$,

(b) $\frac{d^2}{dx^2}[(f*g)(x)]=(f'*g')(x)=(f''*g)(x)$,

(c) $(f*g)^{(m+n)}(x)=\left[f^{(m)}*g^{(n)}\right](x)$,

(d) $\int_{-\infty}^{\infty}(f*g)(x)\,dx=\int_{-\infty}^{\infty}f(u)\,du\int_{-\infty}^{\infty}g(v)\,dv$.

(e) If $g(x)=\frac{1}{2a}H(a-x)$, then
$(f*g)(x)$ is the average value of $f(x)$ in $[x-a,\ x+a]$.

(f) If $G_t(x) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4\kappa t}\right)$, then $(G_t * G_s)(x) = G_{t+s}(x)$.

10. Prove the following results:

(a) $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t - ikx}\, dk = \frac{1}{\sqrt{2t}}\, e^{-x^2/4t}$,

(b) $\int_{-\infty}^{\infty} F(k)\, g(k)\, e^{ikx}\, dk = \int_{-\infty}^{\infty} f(y)\, G(y-x)\, dy$,

(c) $\int_{-\infty}^{\infty} F(k)\, g(k)\, dk = \int_{-\infty}^{\infty} f(y)\, G(y)\, dy$,

(d) $\sin x * e^{-a|x|} = \sqrt{\frac{2}{\pi}}\, \frac{a \sin x}{(1+a^2)}$, (e) $e^{ax} * \chi_{[0,\infty)}(x) = \frac{1}{a}\, \frac{e^{ax}}{\sqrt{2\pi}}$, $a > 0$,

(f) $\frac{1}{\sqrt{2a}} \exp\left(-\frac{x^2}{4a}\right) * \frac{1}{\sqrt{2b}} \exp\left(-\frac{x^2}{4b}\right) = \frac{1}{\sqrt{2(a+b)}} \exp\left(-\frac{x^2}{4(a+b)}\right)$.

11. Determine the solution of the initial-value problem

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \qquad -\infty < x < \infty, \qquad t > 0,\\
u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty.
\end{aligned}
$$

12. Solve

$$
\begin{aligned}
u_t &= u_{xx}, \qquad x > 0, \qquad t > 0,\\
u(x,0) &= f(x), \quad u(0,t) = 0.
\end{aligned}
$$

13. Solve

$$
\begin{aligned}
u_{tt} &= c^2 u_{xxxx} = 0, \qquad -\infty < x < \infty, \qquad t > 0,\\
u(x,0) &= f(x), \qquad u_t(x,0) = 0, \quad -\infty < x < \infty.
\end{aligned}
$$

14. Solve

$$
\begin{aligned}
u_{tt} + c^2 u_{xxxx} &= 0, \qquad x > 0, \qquad t > 0,\\
u(x,0) &= 0, \qquad u_t(x,0) = 0, \qquad x > 0,\\
u(0,t) &= g(t), \quad u_{xx}(0,t) = 0, \qquad t > 0.
\end{aligned}
$$

15. Solve

$$
\begin{aligned}
\phi_{xx} + \phi_{yy} &= 0, \qquad -a < x < a, \quad 0 < y < \infty,\\
\phi_y(x,0) &= \begin{cases} \delta_0, & 0 < |x| < a, \\ 0, & |x| > a. \end{cases}\\
\phi(x,y) &\to 0 \text{ uniformly in } x \text{ as } y \to \infty.
\end{aligned}
$$

16. Solve

$$\begin{aligned} u_t = u_{xx} + t\,u, \quad -\infty < x < \infty, \qquad t > 0, \\ u(x,0) = f(x), \quad u(x,t) \text{ is bounded}, \quad -\infty < x < \infty. \end{aligned}$$

17. Solve

$$\begin{aligned} u_t - u_{xx} + hu = \delta(x)\,\delta(t), \quad -\infty < x < \infty, \qquad t > 0, \\ u(x,0) = 0, \qquad u(x,t) \to 0 \text{ uniformly in } t \text{ as } |x| \to \infty. \end{aligned}$$

18. Solve

$$\begin{aligned} u_t - u_{xx} + h(t)\,u_x &= \delta(x)\,\delta(t), \quad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= 0, \qquad u_x(0,t) = 0, \\ u(x,t) &\to 0 \text{ uniformly in } t \text{ as } x \to \infty. \end{aligned}$$

19. Solve

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < \infty, \qquad 0 < y < \infty, \\ u(x,0) &= f(x), \quad 0 \le x < \infty, \\ u_x(0,y) &= g(y), \quad 0 \le y < \infty, \\ u(x,y) &\to 0 \text{ uniformly in } x \text{ as } x \to \infty \text{ and uniformly in } y \text{ as } x \to \infty. \end{aligned}$$

20. Solve

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad -\infty < x < \infty, \qquad 0 < y < a, \\ u(x,0) &= f(x), \quad u(x,a) = 0, \qquad -\infty < x < \infty, \\ u(x,y) &\to 0 \text{ uniformly in } y \text{ as } |x| \to \infty. \end{aligned}$$

21. Solve

$$\begin{aligned} u_t = u_{xx}, \quad x > 0, \quad t > 0, \\ u(x,0) = 0, \qquad x > 0, \quad u(0,t) = f(t), \quad t > 0, \\ u(x,t) \text{ is bounded for all } x \text{ and } t. \end{aligned}$$

22. Solve

$$\begin{aligned} u_{xx} + u_{yy} = 0, \qquad x > 0, \qquad 0 < y < 1, \\ u(x,0) = f(x), \quad u(x,1) = 0, \qquad x > 0, \\ u(0,y) = 0, \qquad u(x,y) \to 0 \text{ uniformly in } y \text{ as } x \to \infty. \end{aligned}$$

23. Find the Laplace transform of each of the following functions:

(a) t^n, (b) $\cos \omega t$, (c) $\sinh kt$,

(d) $\cosh kt$, (e) te^{at}, (f) $e^{at} \sin \omega t$,

(g) $e^{at} \cos \omega t$, (h) $t \sinh kt$, (i) $t \cosh kt$,

(j) $\sqrt{\frac{1}{t}}$, (k) $\sqrt{t}$, (l) $\frac{\sin at}{t}$.

24. Find the inverse transform of each of the following functions:

(a) $\frac{s}{(s^2+a^2)(s^2+b^2)}$, (b) $\frac{1}{(s^2+a^2)(s^2+b^2)}$,

(c) $\frac{1}{(s-a)(s-b)}$, (d) $\frac{1}{s(s+a)^2}$,

(e) $\frac{1}{s(s+a)}$, (f) $\frac{s^2-a^2}{(s^2+a^2)^2}$.

25. The velocity potential $\phi(x, z, t)$ and the free-surface evaluation $\eta(x, t)$ for surface waves in water of infinite depth satisfy the Laplace equation

$$\phi_{xx} + \phi_{zz} = 0, \qquad -\infty < x < \infty, \qquad -\infty < z \leq 0, \qquad t > 0,$$

with the free-surface, boundary, and initial conditions

$$\begin{aligned}
\phi_z &= \eta_t \quad \text{on} \quad z = 0, \qquad t > 0, \\
\phi_t + g\eta &= 0 \quad \text{on} \quad z = 0, \qquad t > 0, \\
\phi_z &\to 0 \quad \text{as} \quad z \to -\infty, \\
\phi(x, 0, 0) &= 0 \quad \text{and} \quad \eta(x, 0) = f(x), \quad -\infty < x < \infty,
\end{aligned}$$

where g is the constant acceleration due to gravity.
Show that

$$\begin{aligned}
\phi(x, z, t) &= -\frac{\sqrt{g}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^{-\frac{1}{2}} F(k)\, e^{|k|z - ikx} \sin\left(\sqrt{g\,|k|}\, t\right) dk, \\
\eta(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{-ikx} \cos\left(\sqrt{g\,|k|}\, t\right) dk,
\end{aligned}$$

where k represents the Fourier transform variable.
Find the asymptotic solution for $\eta(x, t)$ as $t \to \infty$.

26. Use the Fourier transform method to show that the solution of the one-dimensional Schrödinger equation for a free particle of mass m,

$$\begin{aligned}
i\hbar\psi_t &= -\frac{\hbar^2}{2m}\psi_{xx}, \qquad -\infty < x < \infty, \qquad t > 0, \\
\psi(x, 0) &= f(x), \qquad -\infty < x < \infty,
\end{aligned}$$

where ψ and ψ_x tend to zero as $|x| \to \infty$, and $h = 2\pi\hbar$ is the Planck constant, is given by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, G(x-\xi)\, d\xi,$$

where $G(x,t) = \frac{(1-i)}{2\sqrt{\gamma t}} \exp\left[-\frac{x^2}{4i\gamma t}\right]$ is the Green's function and $\gamma = \frac{\hbar}{2m}$.

27. Prove the following properties of the Laplace convolution:

(a) $f * g = g * f$, (b) $f * (g * h) = (f * g) * h$,

(c) $f * (\alpha g + \beta h) = \alpha (f * g) + \beta (f * h)$, α and β are constants,

(d) $f * 0 = 0 * f$,

(e) $\frac{d}{dt}[(f * g)(t)] = f'(t) * g(t) + f(0)\, g(t)$,

(f) $\frac{d^2}{dt^2}[(f * g)(t)] = f''(t) * g(t) + f'(0)\, g(t) + f(0)\, g'(t)$,

(g) $\frac{d^n}{dt^n}[(f * g)(t)] = f^{(n)}(t) * g(t) + \sum_{k=0}^{n-1} f^{(k)}(0)\, g^{(n-k-1)}(t)$.

28. Obtain the solution of the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \\ u(0,t) &= 0, \qquad u(x,t) \to 0 \text{ uniformly in } t \text{ as } x \to \infty. \end{aligned}$$

29. Solve

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < l, \quad t > 0, \\ u(x,0) &= 0, \qquad u_t(x,0) = 0, \\ u(0,t) &= f(t), \quad u(l,t) = 0, \quad t \ge 0. \end{aligned}$$

30. Solve

$$\begin{aligned} u_t &= \kappa u_{xx}, \quad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= f_0, \qquad 0 < x < \infty, \\ u(0,t) &= f_1, \qquad u(x,t) \to f_0 \text{ uniformly in } t \text{ as } x \to \infty, \quad t > 0. \end{aligned}$$

31. Solve

$$\begin{aligned} u_t &= \kappa u_{xx}, \quad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= x, \qquad x > 0, \\ u(0,t) &= 0, \qquad u(x,t) \to x \text{ uniformly in } t \text{ as } x \to \infty, \quad t > 0. \end{aligned}$$

32. Solve

$$\begin{aligned} u_t &= \kappa u_{xx}, \quad 0 < x < \infty, \qquad t > 0, \\ u(x,0) &= 0, \qquad 0 < x < \infty, \\ u(0,t) &= t^2, \qquad u(x,t) \to 0 \text{ uniformly in } t \text{ as } x \to \infty, \quad t \geq 0. \end{aligned}$$

33. Solve

$$\begin{aligned} u_t &= \kappa u_{xx} - hu, \quad 0 < x < \infty, \quad t > 0, \quad h = \text{constant}, \\ u(x,0) &= f_0, \qquad x > 0, \\ u(0,t) &= 0, \qquad u_x(0,t) \to 0 \text{ uniformly in } t \text{ as } x \to \infty, \quad t > 0. \end{aligned}$$

34. Solve

$$\begin{aligned} u_t &= \kappa u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(x,0) &= 0, \qquad 0 < x < \infty, \\ u(0,t) &= f_0, \qquad u(x,t) \to 0 \text{ uniformly in } t \text{ as } x \to \infty, \quad t > 0. \end{aligned}$$

35. Solve

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(x,0) &= 0, \qquad u_t(x,0) = f_0, \quad 0 < x < \infty, \\ u(0,t) &= 0, \qquad u_x(x,t) \to 0 \text{ uniformly in } t \text{ as } x \to \infty, \quad t > 0. \end{aligned}$$

36. Solve

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \quad 0 < x < \infty, \\ u(0,t) &= 0, \qquad u_x(x,t) \to 0 \text{ uniformly in } t \text{ as } x \to \infty, \quad t > 0. \end{aligned}$$

37. A semi-infinite lossless transmission line has no initial current or potential. A time dependent EMF, $V_0(t)\,H(t)$ is applied at the end $x = 0$. Find the potential $V(x,t)$. Then determine the potential for cases: (i) $V_0(t) = V_0 = \text{constant}$, and (ii) $V_0(t) = V_0 \cos \omega t$.

38. Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation, boundary, and initial conditions are

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2}, \quad z > 0, \qquad t > 0, \\ u(z,t) &= Ut \quad \text{on } z = 0, \qquad t > 0, \\ u(z,t) &\to 0 \quad \text{as} \quad z \to \infty, \qquad t > 0, \\ u(z,t) &= 0 \quad \text{at} \quad t \leq 0, \qquad z > 0. \end{aligned}$$

Explain the implication of the solution.