CH 3: Load and Stress Analysis

Machine elements carry different types of loads (*concentrated, distributed, axial, lateral, moments, torsion, etc.*) according to the function and configuration of each element. These loads cause stresses of different types and magnitudes in different locations in the element.

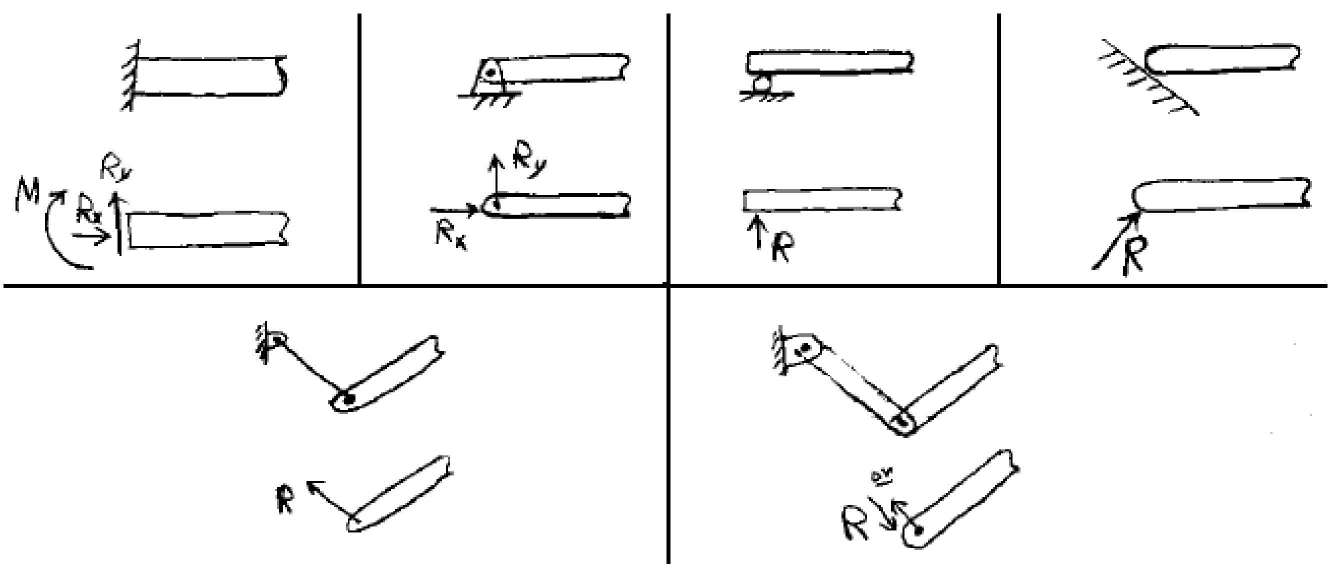
When designing machine elements it is important to locate the critical locations (*or sections*) and to evaluate the stress at the critical sections to ensure the safety and functionality of the machine element.

Equilibrium and Free-Body Diagrams

Equilibrium of a body requires both a balance of forces (to prevent translation) and balance of moments (to prevent rotations).

$$\sum F = 0$$
$$\sum M = 0$$

- A free body diagram (*FBD*) is a sketch of an element or group of connected elements that shows all the forces acting on it (*applied loads, gravity forces, and reactions*)

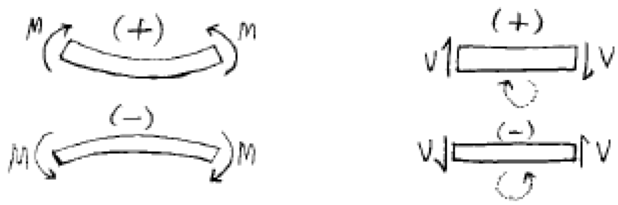


*See **Example 3-1** from text*

Shear and Moment in Beams

Shear and moment diagrams are important in locating the critical sections in a beam (*sections with maximum shear or moment*) such that stresses are evaluated at these sections.

- The sign convection for shear force and bending moment is:

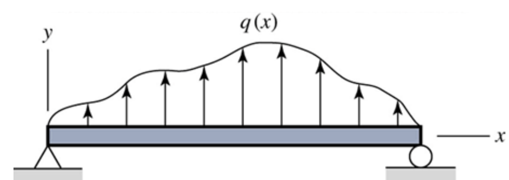


Shear force and bending moment are related by the equation

$$V = \frac{dM}{dx} \qquad \text{(\textit{Shear is the slope of the moment diagram})}$$

When a distributed load $q(x)$ is applied to the beam

$$q = \frac{dV}{dx} = \frac{d^2M}{dx^2}$$



Integrating the relations we get:

$$\int_{x_1}^{x_2} q\,dx = \int_{V_1}^{V_2} dV = V_2 - V_1$$

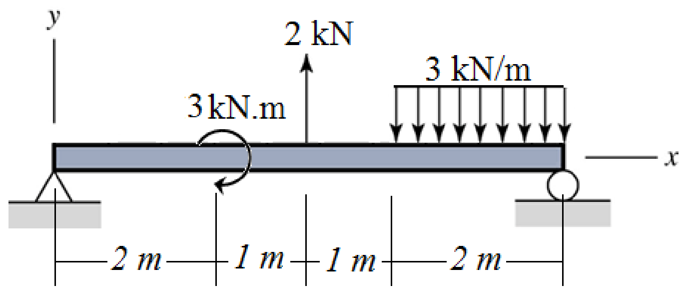
The change in shear force between sections *1 & 2* is equal to the area under the loading diagram

The change in moment between 1 & 2 is equal to the area under the shear diagram

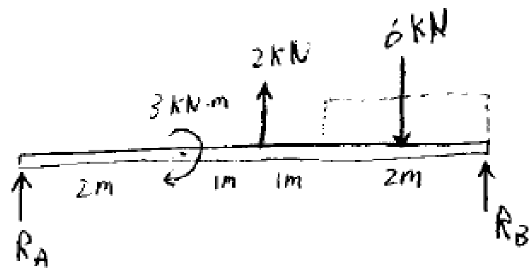
$$\int_{x_1}^{x_2} V\,dx = \int_{M_1}^{M_2} dM = M_2 - M_1$$

- To draw shear and moment diagrams:
  - Draw the *FBD* & find reactions using equilibrium.
  - Make sections and draw their *FBD* and find V & M.

Example: Draw the shear & moment diagrams for the beam shown.



<u>Solution</u>:



$$\circlearrowleft + \sum M_A = 0$$

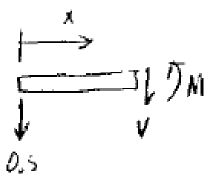
$$-3 + 2(3) - 6(5) + R_B(6) = 0$$

$$\rightarrow R_B = 4.5\ kN$$

$$+\uparrow \sum F_y = 0$$

$$R_A + 2 - 6 + 4.5 = 0$$

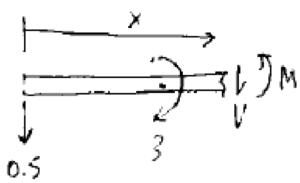
$$\rightarrow R_A = -0.5\ kN = 0.5\ kN \downarrow$$



$\underline{0 < x < 2}$

$\rightarrow V = -0.5\ kN$

$\rightarrow M = -0.5x\ kN.m$

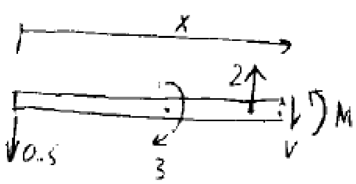


$\underline{2 < x < 3}$

$\rightarrow V = -0.5\ kN$

$M - 3 + 0.5x = 0$

$\rightarrow M = 3 - 0.5x\ kN.m$



$\underline{3 < x < 4}$

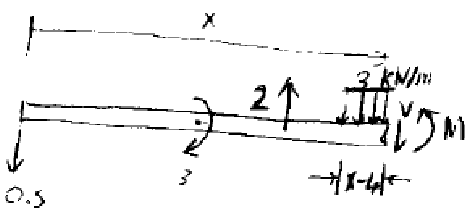
$-0.5 + 2 - V = 0$

$\rightarrow V = 1.5\ kN$

$M - 2(x - 3) - 3 + 0.5x = 0$

$\rightarrow M = 1.5x - 3\ kN.m$

$\underline{4 < x < 6}$



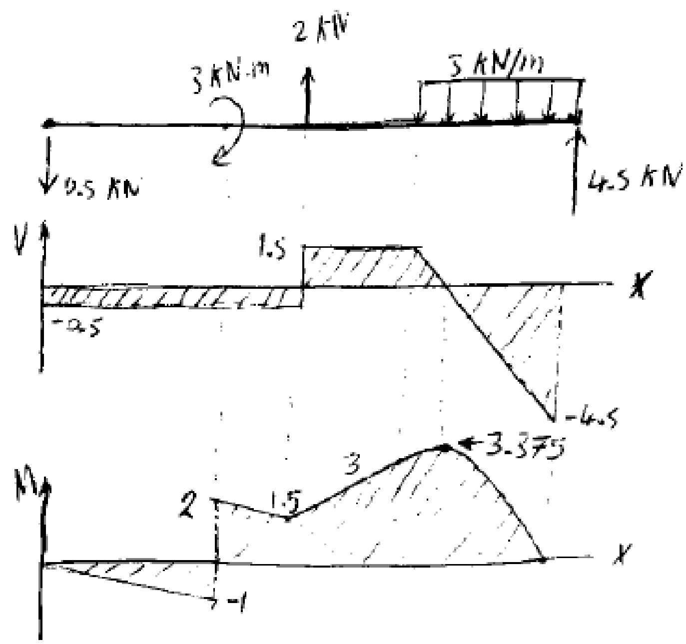
$-0.5 + 2 - 3(x - 4) - V = 0$

$\rightarrow V = -3x + 13.5\ kN$

$$M + 3(x-4)\left[\frac{(x-4)}{2}\right] - 2(x-3)$$

$$-3 + 0.5x = 0$$

$\rightarrow M = -1.5x^2 + 13.5x - 27\ kN.m$



Singularity Functions

When the loading is not simple, obtaining the shear and moment diagrams using sections or integrations become difficult.

The method of singularity functions is used when loading is complicated where it simplifies integration across discontinuities. The advantage of using singularity functions is that it permits writing analytical expressions for shear and moment <u>over a range of discontinuities</u> (*no need for intervals*).

- A singularity functions of x is written as:

$$F_n(x) = < x - a >^n$$

a : Constant distance on the x axis equal to the value of x where the discontinuity occurs.
n : Any integer (positive or negative including zero).

- Rules of the singularity functions:

Evaluation	- $n > 0$	$(x \geq a)$	→ $F_n(x) = (x - a)^n$
		$(x < a)$	→ $F_n(x) = 0$
	- $n = 0$	$(x \geq a)$	→ $F_n(x) = 1$
		$(x < a)$	→ $F_n(x) = 0$
	- $n < 0$	$(x = a)$	→ $F_n(x) = 1$
		$(x \neq a)$	→ $F_n(x) = 0$

- Integration

$n \geq 0$ $$\int < x - a >^n \, dx = \frac{< x-a >^{n+1}}{n+1}$$

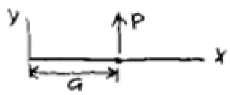
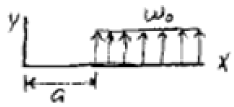
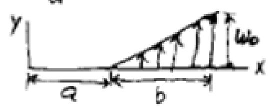
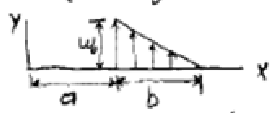
$n < 0$ $$\int < x - a >^n \, dx = < x - a >^{n+1}$$

- Derivation

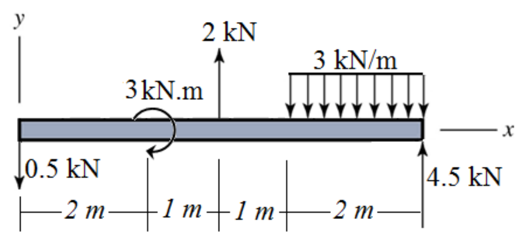
$n \geq 1$ $$\frac{d}{dx} < x - a >^n = n < x - a >^{n-1}$$

$n < 1$ $$\frac{d}{dx} < x - a >^n = < x - a >^{n-1}$$

- The singularity functions for the common types of loading are:

- *Concentrated moment*  $q(x) = M < x - a >^{-2}$
- *Concentrated force*  $q(x) = P < x - a >^{-1}$
- *Step (uniform distributed load)*  $q(x) = w_0 < x - a >^{0}$
- *Ramp*  $q(x) = \frac{w_0}{b} < x - a >^{1}$
- *Inverse Ramp*  $q(x) = w_0 < x - a >^{0} - \frac{w_0}{b} < x - a >^{1}$
- *Parabolic*  $q(x) =< x - a >^{2}$

Example: derive the expressions for loading, shear-force and bending moment for the beam shown.



- Evaluate V & M at $x = 4.5\ m$

<u>Solution</u>:

$$q(x) = -0.5 < x >^{-1} + 3 < x - 2 >^{-2} + 2 < x - 3 >^{-1} - 3 < x - 4 >^{0} + 4.5 < x - 6 >^{-1}$$

$$V = \int q\,dx = -0.5 < x >^{0} + 3 < x - 2 >^{-1} + 2 < x - 3 >^{0} - 3 < x - 4 >^{1} + 4.5 < x - 6 >^{0}$$

$$V(x = 4.5) = -0.5(1) + 3(0) + 2(1) - 3(4.5 - 4)^1 + 4.5(0) = 0\ kN$$

$$M = \int V\,dx = -0.5 < x >^{1} + 3 < x - 2 >^{0} + 2 < x - 3 >^{1} - 1.5 < x - 4 >^{2} + 4.5 < x - 6 >^{1}$$

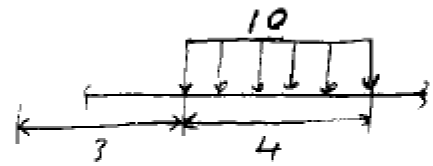
$$M(x = 4.5) = -0.5(4.5) + 3(1) + 2(4.5 - 3)^1 - 1.5(4.5 - 4)^2 + 4.5(0) = 3.375\ kN.m$$

<u>Note</u>: It is not necessary to find the reacitons before using the singularity functions where they can be evaluted from the shear and moment equations by evaluating at $x < 0$ *or* $x > l$ and knowing that both V & $M = 0$ at that value of x.

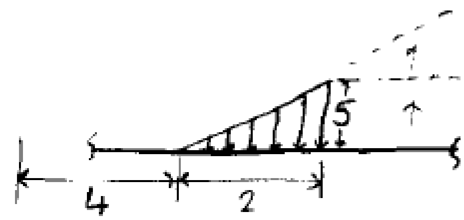
- When there is a <u>distibuted loading</u> that ends <u>before</u> the end of the beam, it needs to be <u>turned-off</u>

Examples:

$$q(x) = \cdots - 10 < x - 3 >^0 \boxed{+ 10 < x - 7 >^0} \ldots$$



$$q(x) = \cdots - \frac{5}{2} < x - 4 >^1 + \frac{5}{2} < x - 6 >^1 + 5 < x - 6 >^0 \ldots$$



Stress

Stress is the term used to define the intensity and direction of the internal forces acting at a given point on a particular plane.

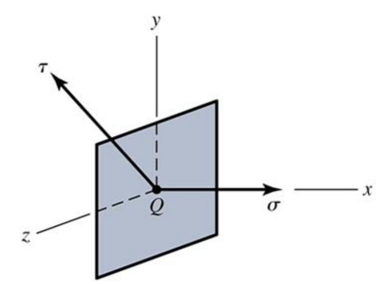
The average stress is defined as force acting over an area

$$\sigma = \frac{P}{A}$$

The stress at a point on a cross-section is thus

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A} = \frac{dP}{dA}$$

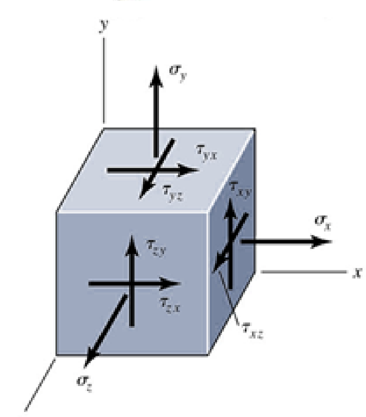
- Which is a vevtor having magnitude and direction



In general, the stress at a point on a cross-section will have components normal and tangential to the surface, which are named as normal steress σ and shear stress τ.

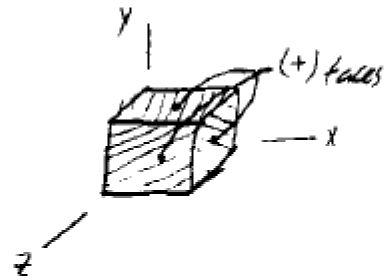
Cartesian Stress Components

The general 3D state of stress at a given point can be shown using a stress element.



- For normal: σ_x — *on the (x) face in the (x) direction*
- For shear: τ_{yz} — *on the (y) face.* — *in the (z) Direction*

- Sign convection
  - Positive stress : *(+) face & (+) direction*
    ***or** (-) face & (-) direction.*
  - Negative stress : *(-) face & (+) direction*
    ***or** (+) face & (-) direction.*

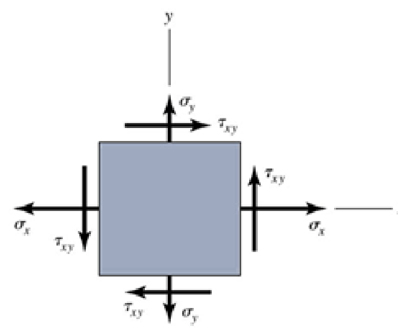


  - Thus, for normal stress tensile stress is positive & compressive stress is negative.
- There are nine stress components, but moment equilibrium requires that:

$$\tau_{xy} = \tau_{yx} \quad , \qquad \tau_{yz} = \tau_{zy} \quad , \qquad \tau_{xz} = \tau_{zx}$$

  - Thus, there are only six independent stress components, three normal and three shear.

- When the stresses on one of the surfaces is zero, the state of stress is called plane stress and the stress components reduce to three: σ_x, σ_y & τ_{xy}.



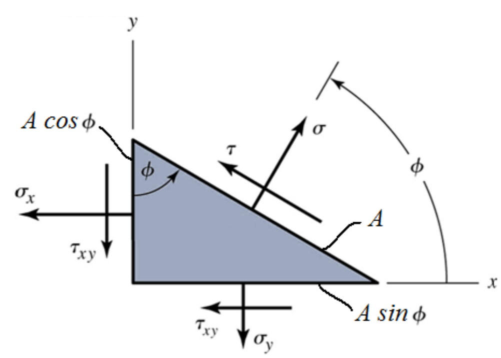
Mohr's Circle for Plane Stress

Consider a wedge shaped element of unit depth subjected to plane stress.

- Equilibrium of forces in the direction of σ requires that:

  This reduces to:

$$\sigma = \sigma_x cos^2\emptyset \; + \; \sigma_y sin^2\emptyset + 2\tau_{xy} \; sin\emptyset cos\emptyset$$



  Using trigonometric identities it reduces to:

$$\boxed{\sigma = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2} cos2\emptyset + \tau_{xy} \; sin2\emptyset} \qquad (1)$$

  Similarly, by summing forces in the τ direction we can get:

$$\boxed{\tau = -\frac{\sigma_x - \sigma_y}{2} sin2\text{Ø} + \tau_{xy} cos2\text{Ø}} \qquad (2)$$

- These two equations are called the plane-stress <u>transformation equations</u>, where they can be used to find the σ & τ in any desired direction defined by an angle Ø (measured from the positive x axis).

To find the <u>maximum and minimum</u> values of stress, we <u>differentiate</u> the σ equitation and set the result <u>equal to zero</u>.

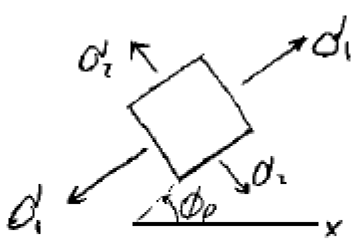
$$\frac{d\sigma}{d\text{Ø}} = -(\sigma_x - \sigma_y) sin2\text{Ø} + 2\,\tau_{xy} cos2\text{Ø} = 0$$

Solving for angle Ø_p we get:

$$\boxed{tan2\text{Ø}_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}}$$

- The angle Ø_p is called the <u>principal angle</u> where its two <u>values</u> define the directions of the <u>max</u> and <u>min</u> normal stresses.

Substituting the values of Ø_p in the σ and τ equations we get:



$$\boxed{\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\boxed{\tau = 0}$$

- At this angle, Ø_p, the normal stresses are maximum (σ_1) and minimum (σ_2) and the shear stress τ = 0.

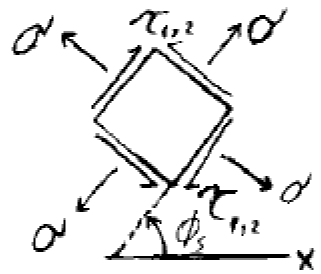
- This direction is called the <u>principal direction</u> and the stresses are called the <u>principal normal stress</u>.

Similarly, finding the angle that defines the direction associated with <u>max</u> and <u>min</u> values of <u>shear stress</u>, we get:

$$\boxed{tan2\text{Ø}_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}}$$

The difference between the max shear stress angle Ø_s and the principal angle Ø_p is 45 degrees

- The principal shear stress and the corresponding value of normal stress are found by substituting $\emptyset_s$ in equations *1* and *2*

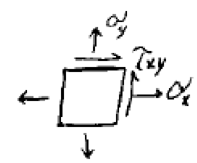


$$\boxed{\tau_{1,2} = \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}.$$

$$\boxed{\sigma = \frac{\sigma_x+\sigma_y}{2}}$$

Equations *1* and *2* define a circle in the σ - τ plane. This circle is known as Mohr's circle, where it provides a convenient method of graphically visualizing the state of stress and it can be used to find the principal stresses as well as performing stress transformation.

- Steps of constructing Mohr's Circle:
  - Given σ_x , σ_y & τ_{xy}
  - Draw the state of stress on a stress element.



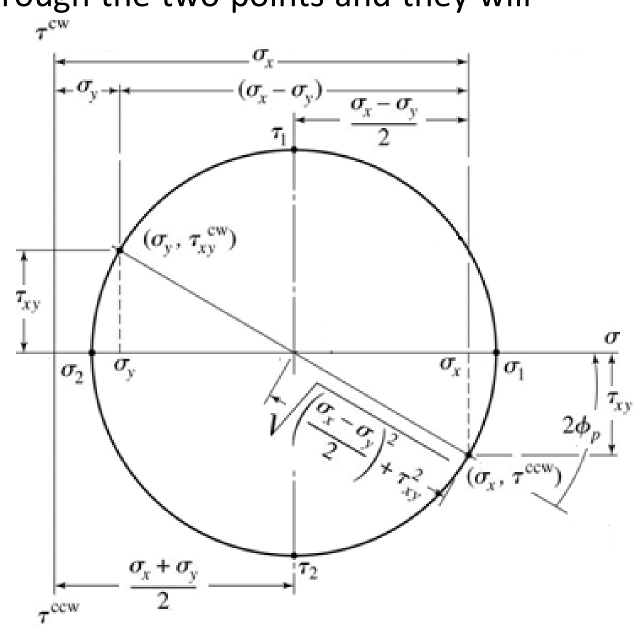
  - Draw the σ and τ axis and locate the center at $\boxed{\left(\frac{\sigma_x+\sigma_y}{2}, 0\right)}$.
  - Locate the two points that define the state of stress
    For the shear stress: if it tends to rotate the element "*CW*" it will be located above the σ axis, and if it tends to rotate "*CCW*" it will be drawn below the σ axis. The circle will pass through the two points and they will be on opposite sides.
  - The radius of the circle is equal to:

$$\boxed{r = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + {\tau_{xy}}^2}}$$

  - The principal normal stresses are located on the intersection points of the circle with σ axis, and they have the values:

$$\boxed{\sigma_{1,2} = center \pm radius}$$

  - The max shear stress is equal to the radius.
  - The direction of the principal axis,



measured from the x direction, is found by determining the angle *2Ø*$_p$ from the circle and rotating in the same direction.

Example: Given the plane stress $\sigma_x = 9\ MPa$, $\sigma_y = 19\ MPa$, $\tau_{xy} = 8\ MPa$

a) Draw Mohr's circle and find the principal normal stress $\sigma_{1,2}$ and the maximum shear stress $\tau_{1,2}$

b) What is the state of stress when the axes are rotated *30˚ CCW*

Solution:

Center: $\sigma_c = \frac{\sigma_x + \sigma_y}{2} = \frac{9+19}{2} = 14\ MPa$

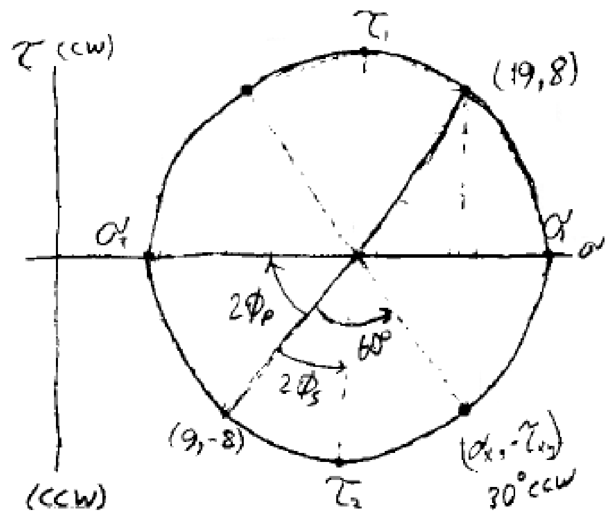
Radius: $r = \sqrt{(14-9)^2 + (8)^2} = 9.43\ MPa$

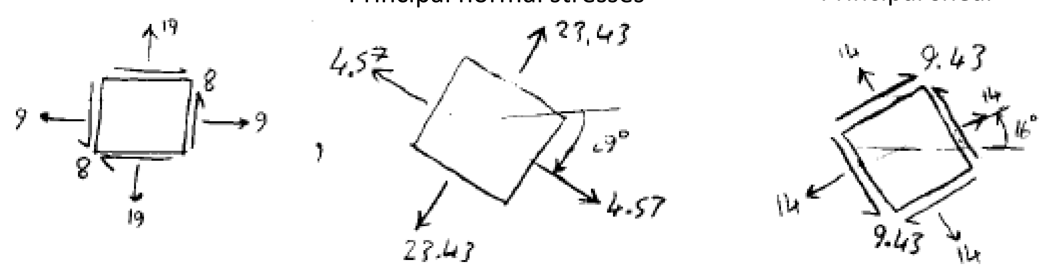
a) $\sigma_{1,2} = 14 \pm 9.43 = 23.43\ ,\ 4.57\ MPa$

$\tau_{1,2} = \pm\ 9.43\ MPa$

$2\emptyset_p = \tan^{-1}\frac{8}{(14-9)} = 58^\circ \quad \rightarrow \emptyset_p = 29^\circ$

$2\emptyset_s = 90 - 58 = 32^\circ \quad \rightarrow \emptyset_s = 16^\circ$



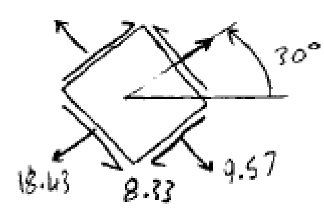


b) Performing a rotation of (*30˚* x *2*) *CCW* from the current state of stress:

$\sigma_x = 14 + 9.43 \cos(180 - 58 - 60) = 18.43\ MPa$

$\sigma_y = 14 - 9.43 \cos(62) = 9.57\ MPa$

$\tau_{xy} = 9.43 \sin(62) = 8.33\ MPa$



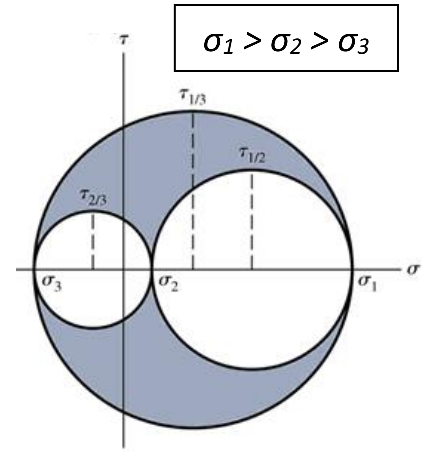
General Three-Dimensional Stress

For the case of 3D stress, there are six components of stress and thus there are three principal normal stress components and three principal shear stresses.

- The three principal stress σ_1 , σ_2 , σ_3 , are found as the three roots of the cubic equation:

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\,\sigma^2 + (\sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{zx})\,\sigma$$
$$-(\sigma_x \sigma_y \sigma_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau^2_{yz} - \sigma_y \tau^2_{zx} - \sigma_z \tau^2_{xy}) = 0$$

$\boxed{\sigma_1 > \sigma_2 > \sigma_3}$



- The principal stresses are labeled such that $\sigma_1 > \sigma_2 > \sigma_3$
- After finding principal stresses the 3D Mohr's Circle can be drawn to help visualizing the state of stress.
- If the principal directions are also needed, tensor notation needs to be used and we find the *Eigen-vectors*.
- The shear stresses are found as:

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2}, \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2}, \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2}$$

  Where the maximum shear stress is $\tau_{1/3}$

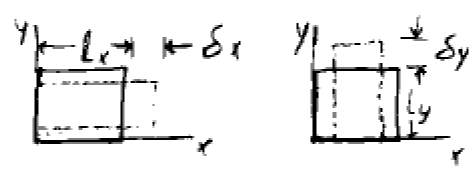
- 3D states of stress are not common in machine elements except for the case of contact stress.
- The 3D Mohr's Circle can be drawn for any state of plane stress knowing that one of the three principal stresses is equal to zero.

Strain

Strain is a non-dimensional measure of the deformation resulting from the stresses acting upon a solid material.

There are two types of strain:

Normal strain $\boldsymbol{\epsilon}$ where it measures the change in length resulting from normal stress.

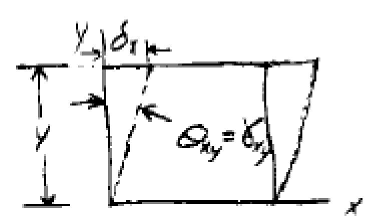


$$\epsilon_x = \frac{\delta_x}{l_x} \quad \& \quad \epsilon_y = \frac{\delta_y}{l_y}$$

Shear strain $\boldsymbol{\gamma}$ where it measures the angular distortion (*the change in angle*) resulting from shear stress.

$$\gamma_{xy} = \frac{\delta_x}{y} = \tan\theta_{yx} \cong \theta_{xy}$$

For small strains



- In the Elastic region under <u>uniaxial stress</u> condition or <u>pure shear</u> stress condition, the stress and strain are related as:

$$\boxed{\sigma = E\epsilon}$$

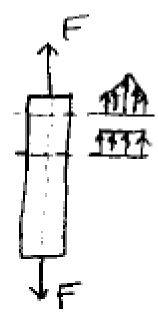
$$v = \frac{-\, lateral\ strain}{axial\ strain}$$

$$\boxed{\tau = G\gamma} \qquad \text{where} \qquad G = \frac{E}{2(1+v)}$$

- For <u>biaxial</u> or <u>triaxial</u> state of stress, these relations are <u>not valid</u> and the generalized hook's law is used to relate stresses and strain.
  - *See the generalized Hook's law equations in text (Eqn. 3-19, page 102).*

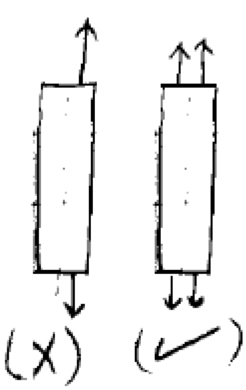
<u>Uniformly Distributed Stress</u>

The assumption of uniformly distributed stress is often made in design when <u>loading is simple</u> such as pure tension, compression or shear.



The section should be taken <u>away</u> from the ends

- For tension or compression

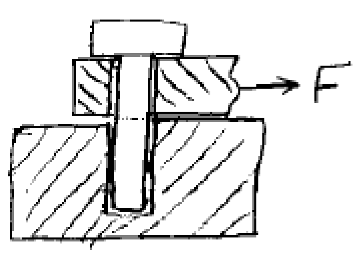


$$\sigma = \frac{F}{A} \qquad \begin{cases} (+) \text{ For Tension} \\ (-) \text{ For compression} \end{cases}$$

* The load should be <u>Centroidal or Symmetric</u>.

- For shear

$$\tau = \frac{F}{A}$$



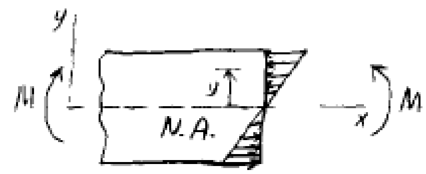
Normal Stress in Beams

Beam bending stress equation (flexure formula) is developed under the following assumptions:

- The beam is straight, long and having a constant cross-section with an axis of symmetry in the plane of bending.
- The material is isotropic, homogeneous, and linearly elastic.
- The beam is subjected to pure bending moment (no axial force, shear or torsion).

The bending stress in beams subjected to bending moment is found as (*see derivation in text*):

$$\boxed{\sigma = -\frac{My}{I}}$$



Where, y: is the height from the neutral axis (centroidal axis)
I: is the moment of inertia about the z axis

The maximum tensile and compressive stresses are at the top and bottom surfaces.

- The maximum bending stress in the beam is usually found using:

$$\boxed{\sigma = \frac{Mc}{I}}$$

where $c = y_{max}$

or sometimes it is written as:

$$\boxed{\sigma = \frac{M}{Z}}$$

where $Z = \frac{I}{c}$ is called the Section Modulus

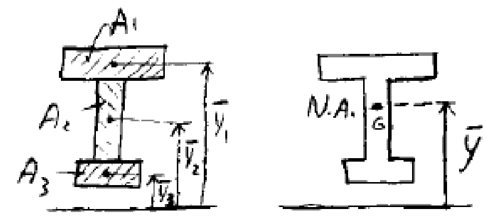
❖ Tables A-6, A-7 and A-8 in the text give the I and z values for some standard cross-section beams.

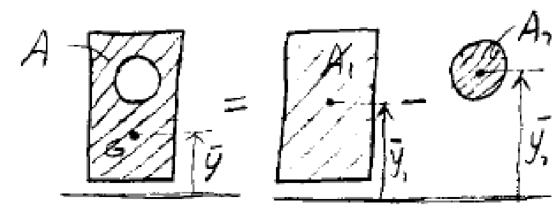
Q: When to use I and when to use Z?

A: in general, it can be said that the Z is more convenient to use when you are designing based on stress while the I is used when you are designing based on deflections.

- Locating the <u>neutral axis</u> and finding the <u>moment of inertia</u> for composite areas (cross-sections):
  - <u>Centroid</u>

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2 + \cdots}{A_1 + A_2 + \cdots}$$

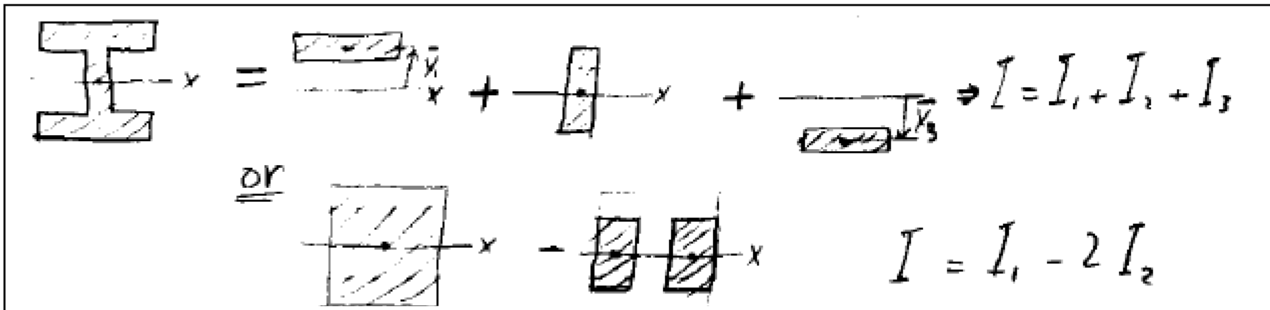




$$\bar{y} = \frac{A_1\bar{y}_1 - A_2\bar{y}_2}{A_1 - A_2}$$

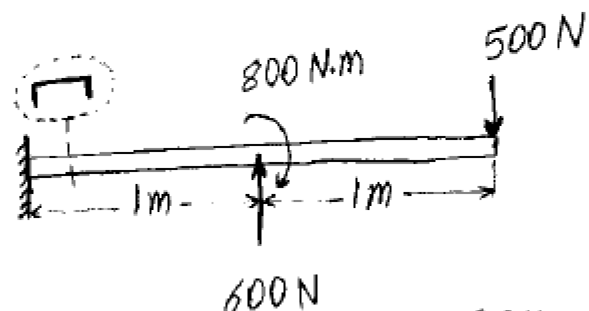
  - <u>Moment of inertia</u>
    When the <u>axis</u> is not passing through the centroid of an area, we use the *<u>parallel axis theorem</u>*.

$$\boxed{I_x = \bar{I}_x + A\bar{y}^2}$$



❖ Table <u>A-18</u> in the text gives the geometric properties of some shapes.

Example: A *C-channel* beam is to be used to support the loading shown. Choose an appropriate standard size from *<u>Table A - 7</u>* such that the maximum bending stress is not to exceed *250 MPa*.

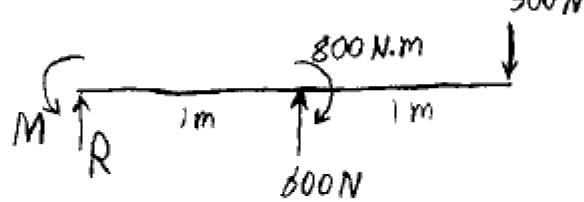


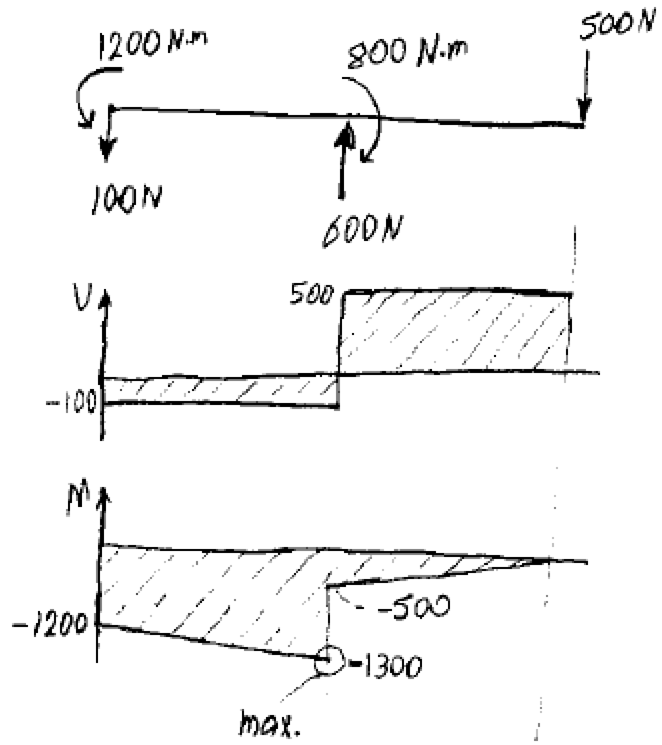
<u>Solution</u>:

$$\sum F_y = 0 \quad \rightarrow \quad R = -100\ N = 100\ N \downarrow$$

$$\sum M = 0 \quad \rightarrow \quad -M + 800 - 600(1) + 500(2) = 0$$

$$\rightarrow \quad M = 1200\ N.m$$





$$\sigma_{max} = \frac{M_{max}}{Z}$$

$$Z = \frac{M_{max}}{\sigma_{max}} = \frac{1300\ (N.m) \times 1000\ (\frac{mm}{m})}{250\ (\frac{N}{mm^2})}$$

➔ $Z = 5200\ mm^3 = 5.2\ cm^3$

with $Z_{2\text{-}2} \geq 5.2\ cm^3$

➔ From Table A-7 choose the beam having *a = 102 mm, b = 51 mm &* $Z_{2\text{-}2} = 8.16\ cm^3$

Shear Stress for Beams in Bending

It is rare to encounter beams subjected to pure bending momet only (no shear). Most beams are subjected to both shear forces and bending moments.

Though the flexure formula (beam bending stress equation) was developed based on the assumption of pure bending moment only, yet it holds reasonably accurate with the presence of shear forces.

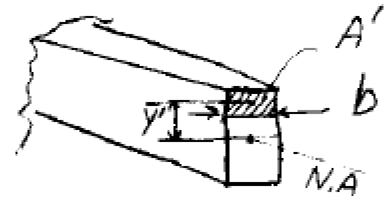
- For a beam subjected to shear force, the shear stress is found as (*see derivation in text*):

$$\boxed{\tau = \frac{VQ}{Ib}}$$

where,

- V: is the shear force at the section of interst.
- Q: is the first moment of inertia *at the height* where τ is determined.
- I: is the section moment of inertia.
- b: is the width at the point where τ is determined.

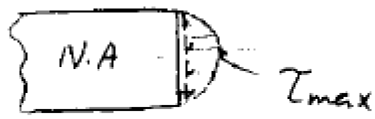
- The first moment of inertia, Q , is found as:



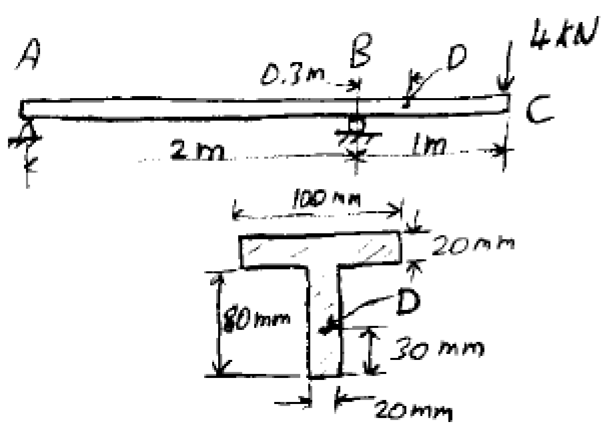
$$\boxed{Q = \bar{y}'A'}$$

- where
  - A' is the area of the portion of the section above or below the point where τ is determined.
  - $\bar{y}'$ is the distance to the centroid of the area A' measured from the neural axis of the beam.

- The shear stress τ is maximum at the neutral axis (*since Q will be max*), and it is zero on the top and bottom surfaces (*since Q is zero*).



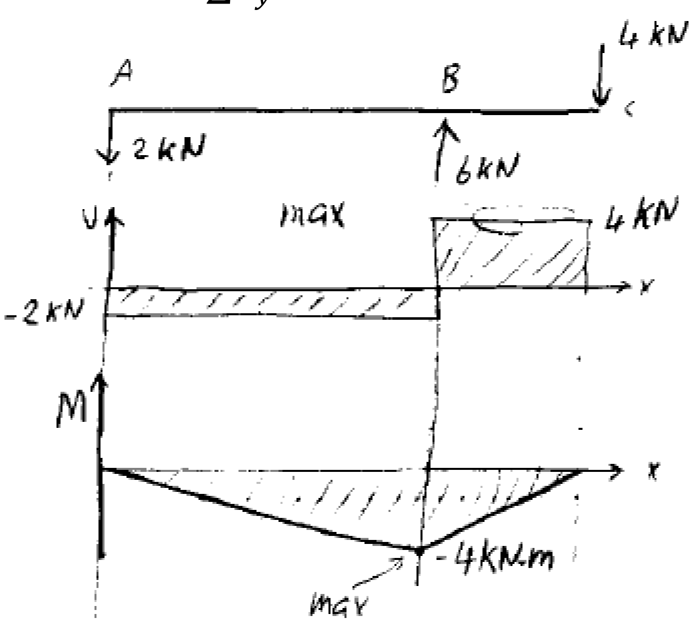
Example: An overhanging simply supported beam having a *T-shaped* cross section carries *4 kN* load as shown. Determine:



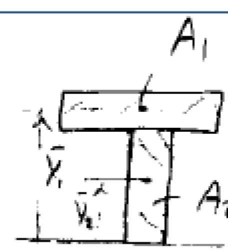
a) The maximum bending stress
b) The maximum shear stress
c) The state of stress at point "*D*"

Solution:

$\sum M_A = 0$ → *R_B = 6 kN*
$\sum F_y = 0$ → *R_A = -2 kN*



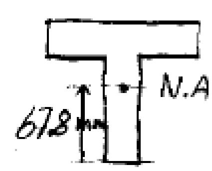
- Neutral Axis:



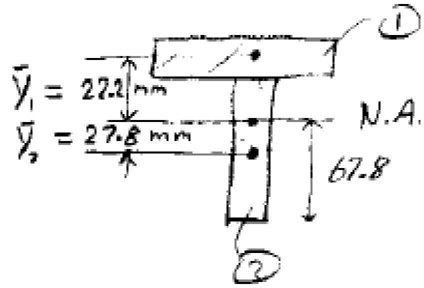
$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

$$\bar{y} = \frac{90(20 \times 100) + 40(80 \times 20)}{(20 \times 100) + (80 \times 20)}$$

➔ $\bar{y} = 67.8\ mm$



- Moment of inertia about the neutral axis:

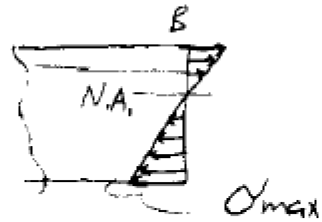


$$I = I_1 + I_2 = (\bar{I}_1 + A_1\bar{y}_1^2) + (\bar{I}_2 + A_2\bar{y}_2^2)$$

$$= \frac{1}{12}(100)(20)^3 + (20 \times 100)(22.2)^2 + \frac{1}{12}(20)(80)^3 + (20 \times 80)(27.8)^2$$

➔ $I = 3.142 \times 10^6\ mm^4$

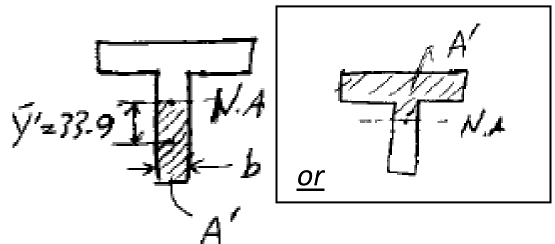
a) Max bending stress on the lower surface at "*B*"



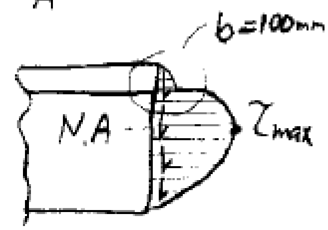
$$\sigma = \frac{Mc}{I} = \frac{(4 \times 10^6)(67.8)}{3.142 \times 10^6} = 86.31\ MPa$$

b) Max shear force between "*B*" and "*C*"

& Max shear stress at the neutral axis



$$Q_{N.A.} = \bar{y}'A' = 33.9(20 \times 67.8) = 45968\ mm^3$$

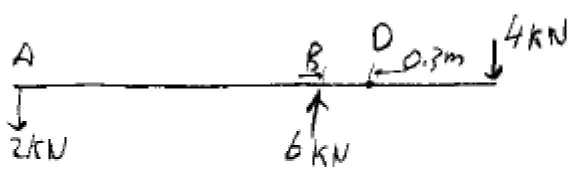


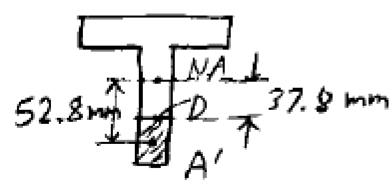
$$\tau_{max} = \frac{VQ_{N.A.}}{Ib} = \frac{(4 \times 10^3)(45968)}{(3.142 \times 10^6)(20)} = 2.93\ MPa$$

c) Stress at "*D*"

$V_D = 4\ kN$

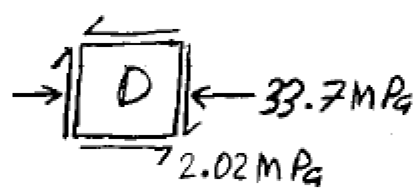
$M_D = -2.8\ kN.m$





$$\sigma_D = -\frac{My}{I} = -\frac{(-2.8 \times 10^6)(-37.8)}{(3.142 \times 10^6)} = -33.7\ MPa$$

$$\tau_D = \frac{VQ}{Ib} = \frac{(4 \times 10^3)(31680)}{(3.142 \times 10^6)(20)} = 2.02\ MPa$$



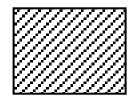
Shear Stress in Standard Section Beams

When desiging a machine element we always look at the maximum stress to ensure the safety of the element.

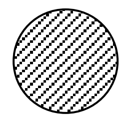
In beams, the maximum shear stress is located at the neutral axis (since Q is max at the N.A.).

- The maximum shear stress for some standard sections is found to be:

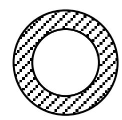
  - Rectangular $\boxed{\tau_{max} = \frac{3V}{2A}}$



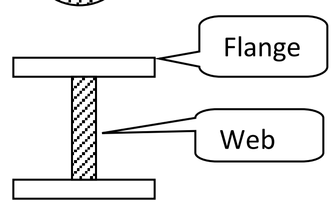
  - Circular $\boxed{\tau_{max} = \frac{4V}{3A}}$



  - Hollow-round (*thin welled*) $\boxed{\tau_{max} = \frac{2V}{A}}$



  - I-beam (*thin welled*) $\boxed{\tau_{max} = \frac{V}{A_{web}}}$

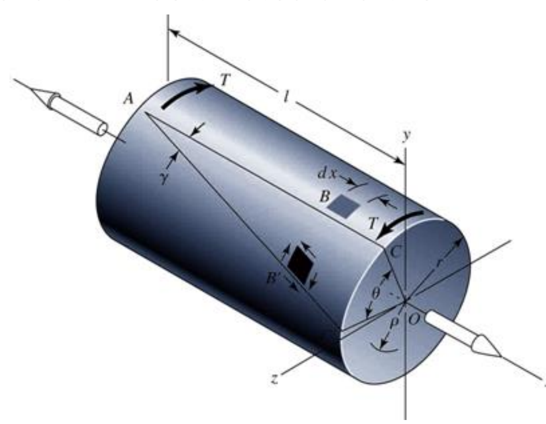


Torsion

When the moment vector is colinear with the axis of an element, it is called a torque vector since it causes the element to be twisted, and the elemet is said to be in torsion.

- When a circular shaft is subjected to torque, the shaft will be twisted and the angle of twist is found to be:

$$\boxed{\theta = \frac{TL}{GJ}}$$



Where
- T: Torque, L: Length, G: Modulus of rigidity $G = \frac{E}{2(1+v)}$ & J: Polar moment of inertia.

  where
  - $J = \frac{\pi}{32} d^4$
  - $J = \frac{\pi}{32}(d_o^4 - d_i^4)$ *for hollow round sections*

- From geometry, the <u>shear strain</u> at any distance from the center (ρ) can be related to the angle of twist as:

$$\gamma = \frac{\rho\theta}{L} \qquad \textit{(assuming small angles)}$$

And maximum shear strain occurs at the <u>outer surface</u>

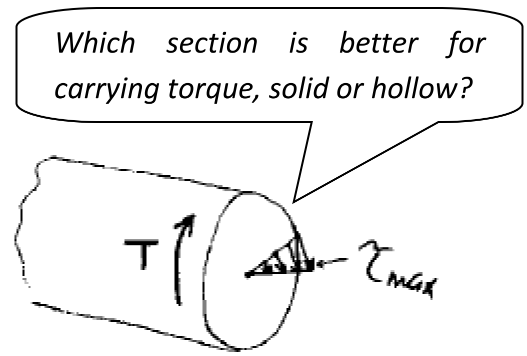
$$\gamma_{max} = \frac{r\theta}{L}$$

- Since shear stress and strain are linearly related in the elastic region ($\tau = G\gamma$), the shear stress at any radius "ρ" is found to be:

$$\boxed{\tau = \frac{T\rho}{J}}$$

And the maximum is at the outer surface,

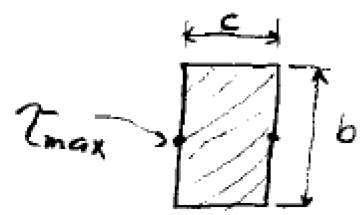
$$\boxed{\tau_{max} = \frac{Tr}{J}}$$



For rectangular cross-sections, the maximum shear stress is found as:

$$\boxed{\tau_{max} = \frac{T}{bC^2}\left(3 + \frac{1.8}{b/c}\right)}$$

where "b" is the longest side



In machin design applications, usually the torue is <u>not given</u>, but rather the transmitted <u>power</u> and rotational speed.

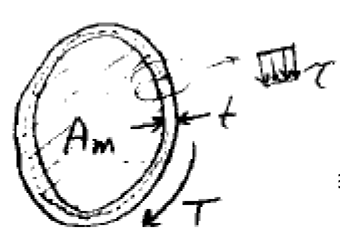
- To find the torque:

$$\boxed{H = T\omega}$$

H: Power ($Watt$)
T: Torque ($N.m$)
ω: Angular velocity (rad/s)

*See **Examples 3-8 & 3-9** from text*

<u>*Torsion of closed thin-walled tubes (t << r)*</u>



For closed thin-walled tubes (*of any shape*) the shear stress is asuumed to be constant through-out thickness.

- The shear stress is found as:

$$\boxed{\tau = \frac{T}{2A_m t}}$$

where A_m is the area enclosed by the section mean line

And, for constant wall thickness, the angle of twist per unit length ($\theta_l = \frac{\theta}{L}$) is found to be:

$$\boxed{\theta_l = \frac{TL_m}{4GA_m^2 t}}$$

where L_m is the perimeter of the section mean line

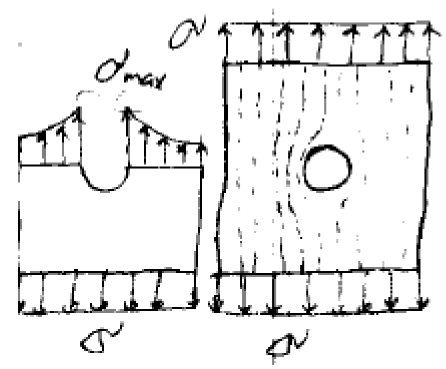
See ***Example 3-10*** *from text*

Stress Concentration

The presence of discontinuities (such as a hole in a plate) alters the stress distribution causing higher stress near the discontinuity. Any type of discontinuity (*hole, shoulder, notch, inclusion*) serve as a stress raiser where it increases the stress in the vicinity of the discontinuity.

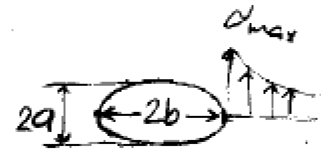
- Stress concentration occures at the region in which stress raisers are present, and a stress concentration factor (K_t) is used to relate the actual maximum stress at the discontinuity to the nominal stress without the discontinuity.

$$\boxed{K_t = \frac{actual\ maximum\ stress}{nominal\ stress} = \frac{\sigma_{max}}{\sigma_o}}$$



- Stress concentration factors are independent of the material properties (as long as the material is in the linear elastic region). They depend only on the type of discontinuity and the geometry.

One of the <u>theoretical</u> stress concentration factors is that of an <u>elliptical</u> hole in an <u>infinite plate</u> loaded in <u>tension</u> which is given as:



$$K_t = 1 + \frac{2b}{a}$$

- Thus, if the hole is <u>circular</u> (a = b) in an <u>infinite</u> plate then $K_t = 3$

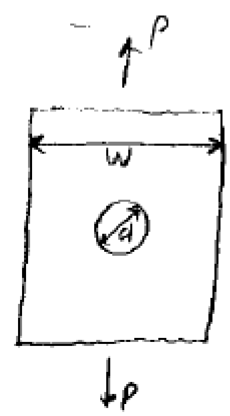
However stress concentration factors are very <u>difficult</u> to find using theoretical analysis, and <u>usually</u> they are found <u>experimentally</u> (using photoelasticity) or using finite element analysis and they are usually presented in <u>charts</u> for different geometric and loading configurations in specialized books (*such as the <u>Peterson's Stress Concentration Factors</u>*).

❖ Tables <u>A-15</u> & <u>A-16</u> in the text give the stress concentration factors for some geometric and loading configurations.

- When using stress concentration factors from charts you should be <u>careful</u> to how K_t is <u>defined</u> (*with respect to stress in the <u>net area</u> or the <u>total area</u>*).
    - $\sigma_o = \frac{P}{wt}$ *Total area*

    or

    - $\sigma_o = \frac{P}{(w-d)t}$ *Net area*



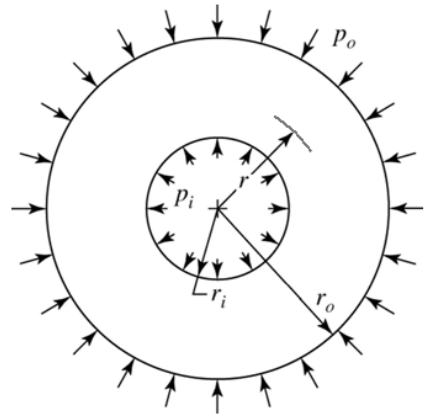
- When dealing with <u>brittle materials</u> it is very <u>important</u> to consider the stress concentrations because rupture will initiate there and the entire part will fail, while for <u>ductile</u> materials stress concentrations are usually *<u>not considered</u>* because the material will yield at the high stress location and this relieves the stress concentration.

<u>Stress in Pressurized Cylinders</u>

Examples of pressurized cylinders include pressure vessels, hydraulic or pneumatic cylinders, gun barrels and pipes carrying high pressure fluids

When there is a pressure difference between the inside and outside of the cylinder, stresses develop in both the radial and tangential directions.

- The tangential and radial stresses (*at any radial distance "r"*) are founds as:

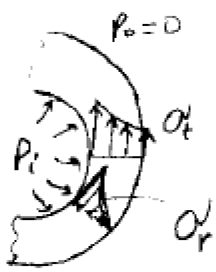


Applicable for both $P_i > P_o$ and $P_o > P_i$

$$\boxed{\sigma_t = \frac{P_i r_i^2 - P_o r_o^2 - r_i^2 r_o^2 (P_o - P_i)/r^2}{r_o^2 - r_i^2}}$$

$$\boxed{\sigma_r = \frac{P_i r_i^2 - P_o r_o^2 + r_i^2 r_o^2 (P_o - P_i)/r^2}{r_o^2 - r_i^2}}$$

- When the external pressure equals zero ($P_o = 0$) the equations reduce to:



$$\boxed{\sigma_t = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)}$$

$$\boxed{\sigma_r = \frac{P_i r_i^2}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)}$$

Where the maximum is at $r = r_i$
$(\sigma_r)_{max} = -P_i$

- If the vessel is close-ended, the longitudinal (*axial*) stress is found as:

$$\boxed{\sigma_l = \frac{P_i r_i^2}{r_o^2 - r_i^2}}$$ *(uniform stress)*

Thin-walled cylinders

When the wall thickness is small compared to the radius ($t \leq \frac{r_i}{20}$), the radial stress is very small and it is assumed to be zero ($\sigma_r = 0$) and the tangential stress is approximately uniform and it is found as:

$$\boxed{\sigma_t = \frac{Pd_i}{2t}}$$

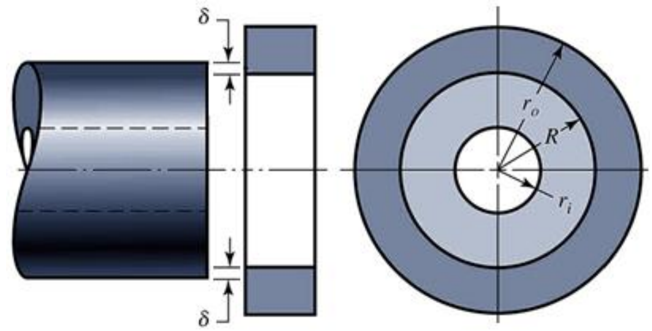
and for close ended vessels the longitudinal stress is:

$$\boxed{\sigma_l = \frac{Pd_i}{4t}}$$

*See **Example 3-14** from text*

Press and Shrink Fits

When two parts are assembled by shrink or press fitting, contact pressure is created at the interface between the two parts.



The magnitude of the contact pressure depends on the amount of interference of the fit where the deformation of the two parts is equal to the interference.

- For two rings made out of the same material, the contact pressure is found to be:

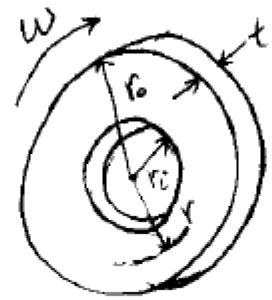
$$\boxed{P = \frac{E\delta}{R}\left[\frac{(r_o^2 - R^2)(R^2 - r_i^2)}{2R^2(r_o^2 - r_i^2)}\right]}$$

where, R: is the common radius and δ: is the radial interference

- The two elements are assumed to have the same length.
  If not (such as a hub that is press fitted on a shaft) an increased pressure is developed at the ends, and a stress concentration factor is used to account for that.

Stress in Rotating Rings

When elements such as flywheels, gears, blowers rotate at high angular velocities, significant stresses develop in the element due to the centrifugal force.



- Such elements are simplified to a rotating ring in order to determine the stresses.
- The stress has two components, tangential and radial. The stresses (*at any radial distance "r"*) are found as:

For constant "t" and $r_o \geq 10t$

$$\boxed{\sigma_t = \rho\omega^2\left(\frac{3+v}{8}\right)\left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3v}{3+v}r^2\right)}$$

$$\boxed{\sigma_r = \rho\omega^2\left(\frac{3+v}{8}\right)\left(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2\right)}$$

Stress initially increases with increasing "r" then it decreases

where, ω: angular velocity and ρ: density

➢ For a rotating disk, use $r_i = 0$

Temperature Effects

When an unrestrained body is subjected to a temperature increase, the body expands and the normal strain is:

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha(\Delta T)$$

where, α: is the coefficient of thermal expansion (*Table 3-3 in text*) and ΔT: is the temperature change.

If the body is restrained in any direction, stress will develop in that direction.

- For a bar with restrained ends, the axial stress is found as:

$$\sigma = -\epsilon E = -\alpha(\Delta T)E$$

Compressive stress

- Similarly , for a plate restrained at all edges, it will have compressive stress in both directions:

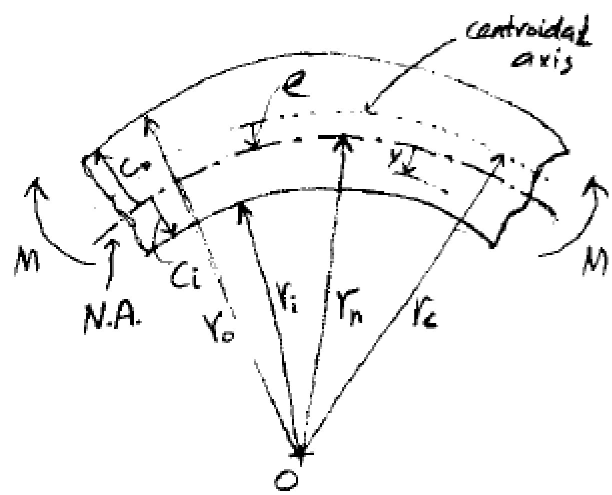
$$\sigma = -\frac{\alpha(\Delta T)E}{1-v}$$

Thermal stresses usually occur during welding or any restrained member subjected to temperature change during operation.

Curved Beams in Bending

When a curved beam (having a symmetric section with respect to the plane of bending) is subjected to bending moment, bending stress will develop in the beam similar to straight beams. However, there are two basic differences:

- The neutral axis does not coincide with the centroidal axis.
- The stress doses not vary linearly from the neutral axis.



(+) M*: Decreases the curvature*
(+) y*: Towards the center of curvature*

- The <u>location of the neutral axis</u> with respect to the center of the curvature is found as:

$$\boxed{r_n = \frac{A}{\int \frac{dA}{r}}}$$

where A: is the cross-sectional area

- The <u>stress at any distance</u> "y" from the neutral axis is found as:

$$\boxed{\sigma = \frac{My}{Ae(r_n - y)}}$$

Stress distribution is hyperbolic

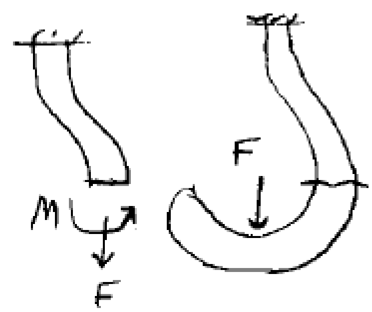
where e: is the distance from the centroidal axis to the neutral axis (*eccentricity*) $e = r_c - r_n$

- The <u>maximum</u> tension and compression stresses occur at the inner and outer surfaces, $y = c_i$ & $y = c_o$

$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad \& \qquad \sigma_o = -\frac{Mc_o}{Aer_o}$$

<u>*Note:*</u> These equations assume pure bending only (*pure moment*).

If the moment is resulting from a force applied to one side of the section, such as the case of a hook, the moment is computed about the <u>centroidal axis</u> <u>*not* the neutral axis</u>, and the additional axial stress is added to the bending stress.



*See **Example 3-15** from text*

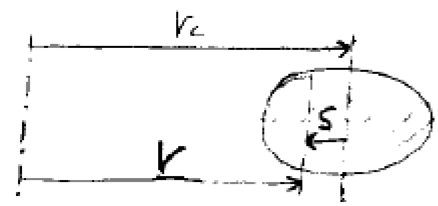
- Note that in the example, the tensile stress at the inner surface is three times greater than the compressive stress at the outer surface, which is not an effective use of the material. Thus, it is better to <u>use more material at the inner radius</u> than at the outer radius (*to reduce the max. stress at the inner surface*). Usually, "T" or unsymmetric "I" cross-sections are used for curved beams in bending.

❖ Table 3-4 gives r_c and r_n values for some of the cross-sections usually used for curved beams.

Approximate Calculations

Since calculating r_n is relatively difficult for complex cross-sections, an approximate solution can be used to find the value of r_n . However, one should be careful since $(e = r_c - r_n)$, a small error in r_n will lead to large error in e (because e is small compared to r_c & r_n) and that will cause a large error in the stress value (since e is in the denominator).

An approximate estimate of the stress can be found as:



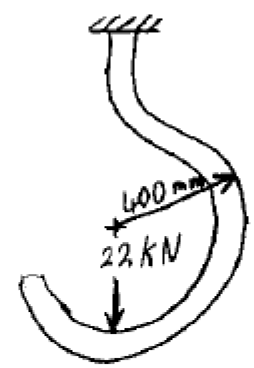
$$e \approx \frac{I}{Ar_c} \quad \rightarrow \quad \boxed{\sigma = \frac{Ms}{I}\frac{r_c}{r}}$$

where “s” is the distance from the centroidal axis.

- This approximate is accurate only when the radius “r_c” is large compared to the beam depth “h”(r >>h).

Example: For the hook in example 3-15, take the outer radius “r_o” to be $400\ mm$.
Find the maximum bending tensile stress at the inner surface using:

a) Exact calculations.
b) Approximate calculations and compute the error.



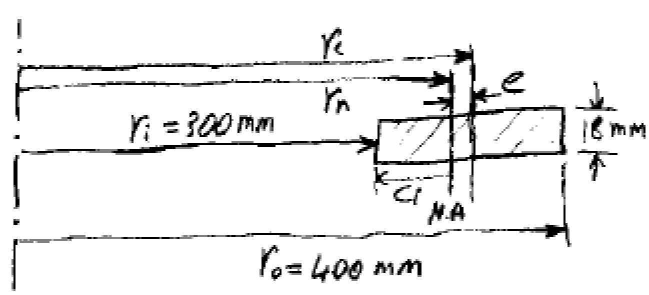
Solution:

a) $$r_n = \frac{h}{ln\frac{r_o}{r_i}} = \frac{100}{ln\frac{400}{300}} = 347.6\ mm$$

$$r_c = 300 + \frac{100}{2} = 350\ mm$$

$$e = r_c - r_n = 2.4\ mm$$

$$A = 100 * 18 = 1800\ mm^2$$



$M = 22000 * 350 = 7.7 \times 10^6 \; N.mm$

$\sigma = \frac{My}{Ae(r_n - y)}$ *Inner surface*: $y = c_i = r_n - r_i$

➔ $\sigma_i = \frac{7.7\times10^6(47.6)}{1800(2.4)(300)} = \boxed{282.8 \; MPa}$

b) $\sigma = \frac{Ms}{I}\frac{r_c}{r}$, $s = 50 \; mm$, *Inner surface*: $r = r_i = 300 \; mm$

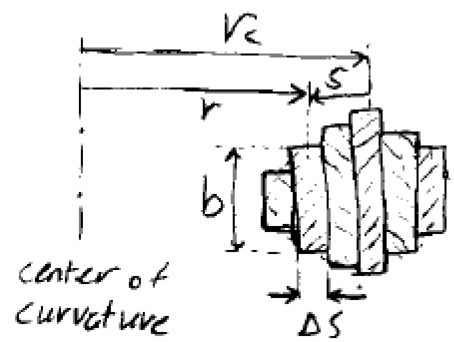
$I = \frac{1}{12}bh^3 = \frac{1}{12}18(100)^3 = 1.5 \times 10^6 \; mm^4$

➔ $\sigma_i = \frac{7.7\times10^6(50)}{1.5\times10^6}\frac{350}{300} = \boxed{299.4 \; MPa}$

$\%Error = \frac{299.4 - 282.8}{282.8} * 100\% = \boxed{5.9 \; \%}$

__Numerical Calculations__

For non-regular cross-sections, r_n can be found using numerical approximation to evaluate the integration by discretizing the cross-sectional-area (*dividing the area into rectangles of small thickness*) where:



$$r_c = \frac{\sum rb\Delta s}{\sum b\Delta s}$$

$$e = \frac{\sum \frac{s}{r_c - s} \; b \; \Delta s}{\sum \frac{b\Delta s}{r_c - s}}$$

$r_n = r_c - e$ *Then continue as usual*

- Note that this method is accurate provided that the section is divided into a <u>sufficient number</u> of rectangles.

Contact Stress

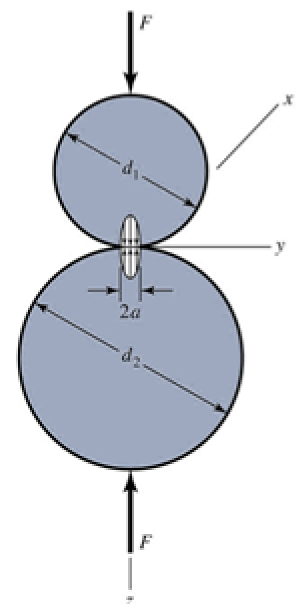
When two bodies having curved surfaces are pressed against each other, the point or line contact changes to area contact. The area of contact depends on the force, the geometry of the two bodies, and the material the bodies are made of. As a result of the contact, three-dimensional state of stress develops at the area of the contact and it might lead to cracking or flaking or similar type of surface failure.

- Examples of contact-stress problems are: wheel on rail, cam and follower, gear teeth contact, etc.

The contact-stress equations were developed by *Hertz* and they are usually called Hertzian stresses.

Spherical Contact

When two spheres of diameters d_1 & d_1 made from two different materials (having E_1, v_1 & E_2, v_2) are pressed against each other with force F, the contact area will be circular with a radius of "a" which is found as:



$$a = \left[\frac{3F}{8}\frac{(1-v_1^2)/E_1 + (1-v_2^2)/E_2}{1/d_1 + 1/d_2}\right]^{1/3}$$

For flat surface: $d = \infty$

For internal surface: d *is negative*

- The resulting pressure distribution is hemispherical and its maximum value is:

$$P_{max} = \frac{3F}{2\pi a^2}$$

And the principal stresses are found as:

Compressive stresses

$$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = -P_{max}\left[\left(1 - \left|\frac{z}{a}\right| tan^{-1}\frac{1}{|z/a|}\right)(1+v) - \frac{1}{2\left(1+\frac{z^2}{a^2}\right)}\right]$$

$$\sigma_3 = \sigma_z = \frac{-P_{max}}{1+\frac{z^2}{a^2}}$$

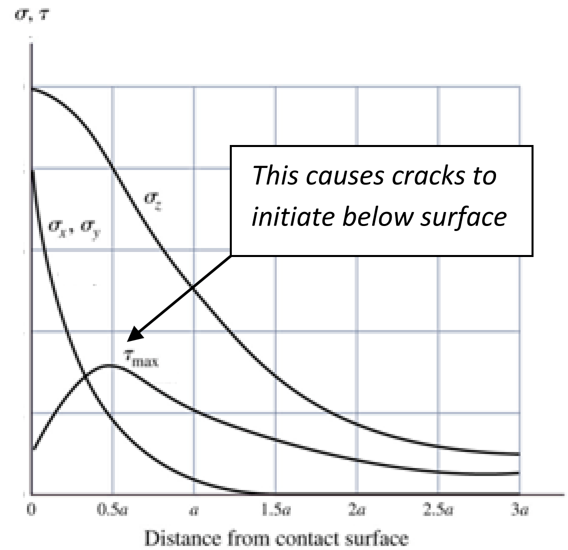
The maximum value of the stress

where Z is the distance from the surface (depth)

For the sphere that is being considered

And the maximum shear stress is:

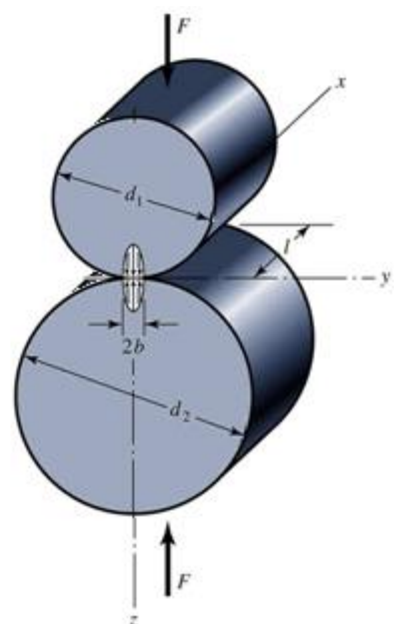
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_2 - \sigma_3}{2}$$



- The principal stresses are maximum at the surface and decrease as depth increases, while the maximum value of shear stress <u>occurs below the surface</u> at $z \cong 0.5a$

<u>*Cylindrical Contact*</u>

For cylinders in contact, the contact area will be rectangular with length "L" and width "$2b$" where:



$$b = \left[\frac{2F}{\pi L} \frac{(1-\upsilon_1^2)/E_1 + (1-\upsilon_2^2)/E_2}{1/d_1 + 1/d_2}\right]^{1/2}$$

And the maximum pressure is:

$$P_{max} = \frac{2F}{\pi b L}$$

And the state of stress is given as:

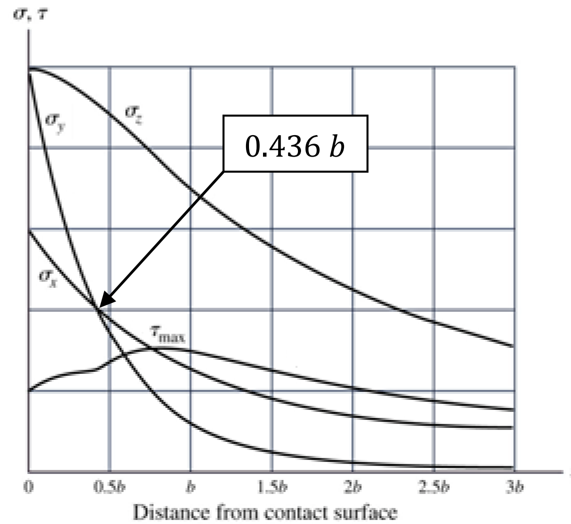
Compressive stresses

$$\begin{cases} \sigma_x = -2\upsilon P_{max}\left(\sqrt[2]{1 + \frac{z^2}{b^2}} - \left|\frac{z}{b}\right|\right) \\ \sigma_y = -P_{max}\left(\frac{1 + 2\frac{z^2}{b^2}}{\sqrt[2]{1 + \frac{z^2}{b^2}}} - 2\left|\frac{z}{b}\right|\right) \\ \sigma_3 = \sigma_z = \frac{-P_{max}}{\sqrt[2]{1 + z^2/b^2}} \end{cases}$$

The maximum value of the stress

And the maximum shear is found as:

$$\tau_{max} = \begin{cases} \frac{\sigma_x - \sigma_z}{2} & for \quad 0 < z < 0.436\ b \\ \frac{\sigma_y - \sigma_z}{2} & for \quad z \geq 0.436\ b \end{cases}$$



Hertz equations are valid if the contact surface is free of shear stress (F only).

- Situations such as gear teeth contact, wheel on rail, etc., involve significant shear force on the contact area and the state of stress can be found more accurately using the "*Smith-Liu* equations".