CH 14: Spur and Helical Gears

This chapter is devoted to analysis and design of spur and helical gears such that they will resist bending failure of teeth and pitting failure of tooth surfaces.

The Lewis Bending Equation

The Lewis equation is used to estimate the bending stress in gear teeth (*max. bending stress at the root of a gear tooth*).

Lewis equation is derived from the basic beam bending stress equation ($\sigma = \frac{My}{I}$):

$$\sigma = K_v \frac{W^t P}{FY}$$

where, *F*: Face width
P: Diametral pitch

- It treats the gear tooth using a factor called "Lewis from factor, Y" (Table 14-2).
- It also includes a correction for dynamic effects "K_v" (*due to rotation of the gear*).

- Lewis equation forms the basis of the *AGMA* bending stress equation used nowadays.

Surface Durability

This section is concerned with the failure of teeth surfaces *(wear)*.

- The most common type of surface failure is pitting which is caused by the repeated high contact stresses.
- An expression for the max contact stress "σ_c" between mating gear teeth can be derived from the *Hertz* equation for two cylinders in contact.
- By adapting the notation used in gearing and including a velocity factor "K_v", the contact-stress can be found as:

$$\sigma_c = -C_p \left[\frac{K_v W^t}{F\cos\emptyset}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right]^{1/2}$$

Compressive stress

where, C_p: *AGMA* elastic coefficient $C_p = \left[\frac{1}{\pi\left(\frac{1-\nu_p^2}{E_p} + \frac{1-\nu_G^2}{E_G}\right)}\right]^{1/2}$

ν_p, ν_G: pinion & gear Poisson's ratio.
E_p, E_G: pinion & gear young's modulus.

W^t: Tangential (*transmitted*) load
F: Face width, $\emptyset$:Pressure angle
r_1& r_2: Instantaneous radii of curvature of the pinion & gear teeth.

- This equation forms the basis of *AGMA* contact stress equation.

AGMA Stress Equations

In the *AGMA* methodology, there are two fundamental stress equations, one for bending stress and another for pitting resistance (*contact stress*).

- Bending stress

$$\sigma = \begin{cases} W^t K_o K_v K_s \dfrac{P_d}{F}\dfrac{K_m K_B}{J} & \textit{US Units} \\[2ex] W^t K_o K_v K_s \dfrac{1}{b m_t}\dfrac{K_H K_B}{Y_J} & \textit{(SI Units)} \end{cases}$$

where:

W^t: Tangential or transmitted load, *lb (N)*
K_o: Overload factor
K_v: Dynamic factor
K_s: Size factor
P_d:Transverse Diametral pitch, "*tooth per inch*"
m_t: Transverse metric module, *(mm)*
F,(b): Face width of the narrower member, *in (mm)*
$K_m, (K_H)$: Load-distribution factor
K_B: Rim thickness factor
$J, (Y_J)$: Geometry factor for bending stress

- Contact stress (*pitting resistance*)

$$\sigma_c = \begin{cases} C_p \sqrt{W^t K_o K_v K_s \dfrac{K_m}{d_p F}\dfrac{C_f}{I}} & \textit{US units} \\[2ex] Z_E \sqrt{W^t K_o K_v K_s \dfrac{K_H}{d_{w1} b}\dfrac{Z_R}{Z_I}} & \textit{(SI units)} \end{cases}$$

where:

$C_p(Z_E)$: *AGMA* elastic coefficient, $\sqrt{psi}$, ($\sqrt{MPa}$)
$C_f(Z_R)$: Surface condition factor
$d_p(d_{w1})$: Pitch diameter of the pinion, *in (mm)*
$I(Z_I)$: Geometry factor for pitting resistance

AGMA Strength Equations

When analyzing gear teeth, after the bending and contact stress values are found, they need to be compared with allowable values of stress (*also called strength*) to make sure the design is satisfactory.

The *AGMA* bending and contact strengths, S_t & S_c, (*i.e., allowable stresses*) are obtained from charts or tables (for different materials) and then are modified by various factors to produce the limiting values of bending and contact stress.

- Allowable (*limiting*) bending stress

$$\boxed{\sigma_{all} = \begin{cases} \dfrac{S_t}{S_F}\dfrac{Y_N}{K_T K_R} & \textit{US units} \\[2ex] \dfrac{\sigma_{FP}}{S_F}\dfrac{Y_N}{Y_\theta Y_Z} & \textit{(SI units)} \end{cases}}$$

where,

$S_t(\sigma_{FP})$: *AGMA* bending strength, *psi (MPa)*
Y_N: Stress cycle life factor for bending
$K_T(Y_\theta)$: Temperature factor
$K_R(Y_Z)$: Reliability factor
S_F: The *AGMA* factor of safety

- Allowable (*limiting*) contact stress

$$\boxed{\sigma_{c,all} = \begin{cases} \dfrac{S_c}{S_H}\dfrac{Z_N C_H}{K_T K_R} & \textit{US units} \\[2ex] \dfrac{\sigma_{HP}}{S_H}\dfrac{Z_N Z_W}{Y_\theta Y_Z} & \textit{(SI units)} \end{cases}}$$

where:

$S_C(\sigma_{HP})$: *AGMA* contact strength, *psi (MPa)*
Z_N: Stress cycle life factor for pitting
$C_H(Z_W)$: Hardness ratio factor for pitting resistance *(only for the gear)*
S_H: *AGMA* factor of safety

❖ The *AGMA* bending strength (S_t) values are given in *Figures 14-2, 14-3, 14-4* and *Tables 14-3, 14-4* *(note that it is termed as "allowable bending stress numbers").*

- ❖ The *AGMA* contact strength (S_c) values are given in *Figure 14-5* and *Tables 14-5, 14-6, 14-7* (*note that it is termed as "allowable contact stress numbers").*
  - ➢ The values given in *AGMA* charts and tables are based on:
    - Unidirectional loading
    - *10*^{7} stress cycles
    - *99* percent reliability
  - ▪ When two-way loading occurs, such as in idler gears, *AGMA* recommends multiplying the bending strength (S_t) value by *0.7.*
    - o *But this is not used for the* S_C *value, why?*

two-way loading
(fully reversed loading cycles on the teeth)

Geometry Factors, J and I (Z_J and Z_I)

Geometry factors are used to account for the tooth form in the stress equations.

- • Bending-stress geometry factor, J (Y_J).
  - ▪ This factor depends on the shape of the tooth and the distance from the tooth root to the highest point of single-tooth contact.
  - ▪ It also includes the effect of stress concentration in the tooth and the ratio of face width upon which load is applied (*i.e., the length of line of contact in helical gears*).
- ❖ The value of J for spur gears with 20° pressure angle and full-depth teeth is found from *Fig. 14-6*.
- ❖ The value of J for helical gears with 20° normal pressure angle is found from *Figs. 14-7 & 14-8*.

- • Contact-stress geometry factor, I (Y_I)

  Also called by *AGMA* as the *pitting-resistance* geometry factor.
  - ▪ It accounts for the values of the instantaneous radius of curvature of the two teeth at the point of contact (*and for the length of contact line for "helical gears"*).
  - ▪ Its value can be found as:

$$\boxed{I = \begin{cases} \dfrac{\cos\phi_t \sin\phi_t}{2m_N}\dfrac{m_G}{m_G+1} & \text{External Gears} \\[2ex] \dfrac{\cos\phi_t \sin\phi_t}{2m_N}\dfrac{m_G}{m_G-1} & \text{Internal Gears} \end{cases}}$$

where:

ϕ_t: Pressure angle for spur gears

or Transverse pressure angle for helical gears.

m_G: Speed ratio, $m_G = \frac{N_G}{N_P} = \frac{d_G}{d_P}$

m_N: Load-sharing ratio

- $m_N = 1$ *for Spur gears*
- $m_N = \frac{p_N}{0.95Z}$ *for Helical gears*

where;

Normal circular pitch measured on the base circle

Normal circular pitch

"p_N" is the normal base pitch $p_N = p_n \cos \emptyset_n$

"Z" is the length of line of action in the transverse plane,

$$Z = \left[\left(r_p + a\right)^2 - r_{bp}^2\right]^{1/2} + [(r_G + a)^2 - r_{bG}^2]^{1/2} - \left(r_p + r_G\right) \sin \emptyset_t$$

- Where "r_p & r_G" are the pitch radii of pinion and gear, "a" is the addendum and "r_{bp}& r_{bG}" are the base-circle radii of pinion and gear.

Remember that $r_b = r \cos \emptyset_t$

- *Note:* in the "Z" equation, if any of the first two terms is larger than the third term, then it should be replaced by the third term (i .e., they will cancel each other).

The Elastic Coefficient, $C_p(Z_E)$

The coefficient combines the elastic constants of the gear and pinion.

- The value of " C_p" (Z_E) can be found as:

$$\boxed{C_p = \sqrt{\frac{1}{\pi\left(\frac{1 - v_p^2}{E_P} + \frac{1 - v_G^2}{E_G}\right)}}}$$

❖ Or "*easier*" from *Table 14-8*.

Dynamic Factor, K_v

This factor is used to account for inaccuracies in the manufacture and meshing of gear teeth in action, which cause deviation from the uniform angular speed a gear pair is supposed to have.

- *AGMA* uses a “*transmission accuracy-level numbers*”,Q_v , to quantify gears into different classes according to manufacturing accuracy (*tolerances*).
  - Q_v From 3 to 7 is for commercial quality gears.
  - Q_v From 8 to 12 is for precision quality gears.

❖ The value of K_v can be found using *Eqn.(14-27)* in the text or from *Fig. 14-9* where it gives K_v as a function of pitch-line speed for different Q_v classes.

Overload Factor, K_o

This factor is used to account for external loads exceeding the nominal tangential load W^t (*such as variations in torque due to firing of cylinders in internal combustion engines*).

- The values of K_o are based on field experience in a particular application.

❖ Values of K_o can be found from the table given in *Fig. 14-17.*

Size Factor, K_s

This factor is used to account for non-uniformity of material properties due to size.

- Standard size factors for gear teeth have not been established yet, thus, *AGMA* suggests using $K_s = 1$ (*if the size effect is known,* $K_s > 1$).
- However, from the formulation given in *Chapter 7* for the endurance limit size correction factor, an expression can developed for computing *size* factor “K_s” for gear teeth which is:

$$\boxed{K_s = 1.192\left(\frac{F\sqrt{Y}}{P}\right)^{0.0535}}$$

where, *F*: Face width
P: Diametral pitch
Y: Lewis from factor (Table 14-2)

- *Note:* If K_s was found from the equation to be less than one, then we will use $K_s = 1$.

Surface Condition Factor, $C_f(Z_R)$

This factor depends on surface finish, residual stress and work hardening.

- It is used only in the contact stress equation.
- Standard surface conditions for teeth surface are not yet defined. Thus, we will use $C_f = 1$.

Load-Distribution Factor, $K_m(K_H)$

This factor is used to account for the non-uniform load distribution along the line of contact.

- One of the causes for non-uniform load distribution is the misalignment of the gear axis resulting from the deformation of the shafts carrying the gears. Other reasons include the inaccuracy in manufacturing and assembly.
- The load-distribution factor can be found as:

$$\boxed{K_m = 1 + C_{mc}(C_{pf}C_{pm} + C_{ma}C_e)}$$

Note that this is valid only when:
- ✓ $F/d_p \leq 2$
- ✓ $F \leq 40\ in$
- ✓ Gears mounted between two bearings
- ✓ Contact across full width of narrowest gear

where,

Crowning

$$C_{mc} = \begin{cases} 1 & \text{for non-crowned teeth} \\ 0.8 & \text{for crowned teeth} \end{cases}$$

Face width (misalignment magnification)

$$C_{pf} = \begin{cases} F/10d - 0.025 & for\ F \leq 1\ in \\ F/10d - 0.0375 + 0.0125F & for\ 1 < F \leq 17\ in \\ F/10d - 0.1109 + 0.0207F - 0.000228F^2 & for\ 17 < F \leq 40\ in \end{cases}$$

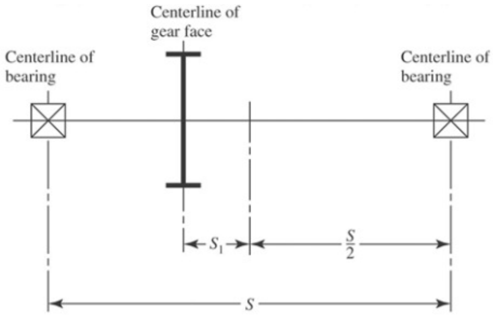
To be conservative we use d_p

➢ *Note: when $F/10d < 0.05$, use $F/10d = 0.05$ instead*

Mounting position

$$C_{pm} = \begin{cases} 1 & \text{for straddle-mounted pinion with } S_1/S < 0.175 \\ 1.1 & \text{for straddle-mounted pinion with } S_1/S \geq 0.175 \end{cases}$$

- *See the figure for S & S_1*
  S: full span
  S_1: distance from midspan



Manufacturing accuracy

$$C_{ma} = A + BF + CF^2$$

❖ Values of *A, B, & C* are found in *Table 14-9*

❖ C_{ma} can also be found from *Fig. 14-11*

Extra adjustment

$$C_e = \begin{cases} 0.8 & \text{for gearing adjusted at assembly, or compatibility improved by lapping, or both} \\ 1 & \text{for all other conditions} \end{cases}$$

Rim-Thickness Factor, K_B

When the rim-thickness is not sufficient, it will not provide full support for the tooth causing increased stress.

- The rim-thickness factor is used to account for the increase in bending stress in thin-rimmed gears.

❖ The value of K_B depends on the rim-thickness to tooth-height ratio, and it can be found from *Figure 14-16*.

Hardness-Ratio Factor, C_H

The pinion has less number of teeth than the gear and therefore the teeth of the pinion are subjected to more cycles of contact stress. To compensate for that, different heat treatments are used for the pinion and the gear to make the pinion harder than the gear.

- The hardness-ratio factor is used to account for the difference in hardness, and it is used *only for the gear*.

❖ For through-hardened pinion and gear, the C_H value can be found from the *Figure 14-12* (*for* $1.2 \leq {}^{H_{Bp}}\!/_{H_{BG}} \leq 1.7$), or from *Eqn. 14-36* in text.

where H_{Bp} & H_{BG} are *Brinell* hardness for pinion and gear

- If ${}^{H_{Bp}}\!/_{H_{BG}} < 1.2$, $C_H = 1$
- If ${}^{H_{Bp}}\!/_{H_{BG}} > 1.7$, $C_H = 1 + 0.00698\,(m_G - 1)$ where m_G is the speed ratio

❖ For surface-hardened pinion (with hardness of *Rockwell-C*: *48* or harder) run with through-hardened gear (*180 to 400 Brinell*), the C_H value can be found from *Figure 14-13* as a function of the pinion surface finish "f_p".

Stress Cycle Life Factors, Y_N *and* Z_N

The *AGMA* bending strength "S_t" and contact strength "S_c" are based on 10^7 load cycles.

- The load-cycle factors Y_N and Z_N, are used to modify the *AGMA* strength for lives other than 10^7 cycles.

❖ The values of Y_N and Z_N are found from *Figures 14-14 & 14-15* according to the number of load cycles for each gear.

  - ○ Note that for 10^7 Cycles $Y_N = Z_N = 1$

Temperature Factor, $K_T(Y_\theta)$

This factor is used to modify the *AGMA* strengths for the effect of high operating temperatures.

- For lubricant (or gear-blank) temperatures up to *250°F* (*120°C*): K_T=*1*
- For temperature higher that *250°F*, K_T will be greater than one. But no data is available for such conditions.
- Heat exchangers may be used to maintain temperature below *250°F*

Reliability Factor, $K_R(Y_Z)$

The *AGMA* strengths S_t & S_c are based on *0.99* reliability.

- The reliability factor is used to modify the *AGMA* strength for reliabilities other that *0.99*.

❖ The values of K_R for some reliability values are found in *Table 14-10*.

- For reliability values other than those given in the table, the K_R value can be found as:

$$K_R = \begin{cases} 0.658 - 0.0759\ln(1-R) & \textit{for } 0.5 < R < 0.99 \\ 0.5 - 0.109\ln(1-R) & \textit{for } 0.99 < R < 0.9999 \end{cases}$$

Safety Factors, S_F *and* S_H

A factor of safety is used to account for unquantifiable elements affecting the stress.

- When designing gear stets, a factor of safety becomes a design factor (*i.e., specified by the designer*) indicating the desired strength-to-stress ratio.
- When analyzing or doing a design assessment for a gear set, the value of safety factor is the ratio of strength to stress.
  - Bending stress factor of safety S_F is found as:

$$S_F = \frac{S_t Y_N/(K_T K_R)}{\sigma} = \frac{\textit{Fully corrected bending strength}}{\textit{Fully corrected bending stress}}$$

  - Where S_F is linearly related to the transmitted load W^t (*since the relation between* σ *and* W^t *is linear*).

  - Contact stress factor of safety S_H is found as:

$$S_H = \frac{S_c Z_N \overset{\text{only for the gear}}{\overset{\downarrow}{C_H}}/(K_T K_R)}{\sigma_c} = \frac{\textit{Fully corrected contact strength}}{\textit{Fully corrected contact stress}}$$

- Where S_H is not linearly related to the transmitted load W^t (*since the relation between* σ_c*and* W^t *is not linear*).

- Because the difference in the relation of S_F and S_H with the transmitted load, if we want to compare the values of S_F and S_H in an analysis (in order to determine the nature and severity of the threat of failure), then we should:
  - Compare S_F with S_H^2 *for linear or helical contact*
  - Compare S_F with S_H^3 *for spherical contact (crowned teeth)*

Analysis

- Figures 14-17 and 14-18 give a "road map" listing the *AGMA* equations for determining the bending and contact stresses and strengths as well as the factors of safety.
- It should be clear that most of the terms in the bending and contact stress or strength equations will have the same value for the pinion and the gear.
- The factors that could have two different values for the pinion and gear are: K_S, J, K_B, S_t, S_c Y_N, Z_N.

See ***Example 14-4*** *from text*

- In such example, the factor of safety of the entire "gear set" is the lowest among all factors of safety, which is in this case the wear factor of safety for the gear $(S_H)_G^2$*=2.31*
- When the factor of safety is larger than one, this means that the performance of the gear set exceeds the requirements (*i.e., the gear set will run with the specified load for a longer life*).
- A safety factor of, for example *2.31*, for the gear set means that we can, theoretically, increase the transmitted load by *2.31* times and still get the required performance.
- Optimal design is obtained when all the different factors of safety are equal, however, it is preferable to have bending factors of safety that are slightly higher than the wear factors of safety because bending failure (teeth breakage) is more dangerous than surface failure (wear).
- The wear resistance of gears can be controlled by controlling the surface hardness.
- The bending performance of gears can be controlled by controlling the core hardness.

- Both bending and wear factors of safety are influenced by the tooth size (*face width & diametral pitch, thus gears diameter*) but their influence on bending stress is greater than that on contact stress.

*See **Example 14-5** from text*

- It is desirable to make the bending factors of safety for the pinion and gear, equal. This can be done by controlling the "*core*" hardness (*and thus bending strength*) of the pinion and gear.
  - The bending factors of safety of the pinion and gear are,

$$(S_F)_P = \left(\frac{\sigma_{all}}{\sigma}\right)_P = \left(\frac{S_t Y_N/(K_T K_R)}{W^t K_o K_v K_s \frac{P_d}{F}\frac{K_m K_B}{J}}\right)_P, (S_F)_G = \left(\frac{\sigma_{all}}{\sigma}\right)_G = \left(\frac{S_t Y_N/(K_T K_R)}{W^t K_o K_v K_s \frac{P_d}{F}\frac{K_m K_B}{J}}\right)_G$$

  - Equating both factors of safety and canceling identical terms (*including K_s which is almost equal*) and solving for $(S_t)_G$

$$(S_t)_G = (S_t)_P \frac{(Y_N)_P}{(Y_N)_G}\frac{J_P}{J_G}$$

  - Knowing that $Y_N = \alpha N^\beta$, we can write $(Y_N)_P = \alpha N_P^\beta$
    and $(Y_N)_G = \alpha (N_P/m_G)^\beta$
  - Substituting and simplifying we get,

$$\boxed{(S_t)_G = (S_t)_P m_G^\beta \frac{J_P}{J_G}}$$

The gear can be less strong than the pinion for the same factor of safety

- Similarly, the contact-stress factors of safety for the pinion and gear can be made equal by controlling the "surface" hardness (*and thus contact strength*).
  The relation between contact-strengths for pinion and gear can be found to be,

$$\boxed{(S_c)_G = (S_c)_P m_G^\beta}$$

*See **Examples 14-6 & 14-7** from text*

Design of Gear Mesh

The decisions needed when designing a gear set are divided in two categories:

Priori decisions

- Function: load, speed, reliability, life, K_o
- Unquantifiable risk: design factor n_d ← Design factor of safety
- Tooth system: ϕ, ψ, a, b, root fillet radius
- Gear ratio m_G: N_P, N_G
- Quality number: Q_V

Design decisions
(Total of eight decisions)

- Diametral pitch, P_d } Gear Size
- Face width, *F* } Gear Size
- Pinion material, core hardness, case hardness } Gear Strength
- Gear material, core hardness, case hardness } Gear Strength

- When designing, some iteration will be required until a satisfactory design is reached. Thus, it is important to place the design decisions in order of importance (*i.e., impact on the amount of work to be redone in iterations*).

The suggested design strategy is as follows:

1. Select a trial Diametral pitch, P
2. Take the face width to be $F = 4\pi/P$ (*face width should be within* $3\pi/P \le F \le 5\pi/P$)
3. Start with bending analysis
   - 3.1 Pinion: find the bending stress σ (take W^t as $n_d\ W^t$)
     - 3.1.1 Choose a material and core hardness
     - 3.1.2 Solve for, F , such that $\sigma = \sigma_{all}$(if *F* is not within range return to 3.1.1 or to 1)
     - 3.1.3 Choose a value for, F, slightly larger than the calculated value & check the factor of safety S_F
   - 3.2 Gear: Find necessary material strength S_t such that $(S_F)_G = (S_F)_P$
     - 3.2.1 Choose a material and core hardness
     - 3.2.2 Find stress σ, then check bending factor of safety S_F
4. Start wear analysis
   - 4.1 Pinion: find contact stress σ_c (take W^t as $n_d\ W^t$)

    - 4.1.1 Find S_c such that $\sigma_c = \sigma_{c,all}$
    - 4.1.2 Find required case hardness & choose larger harness
    - 4.1.3 Check wear factor of safety S_H^2
- 4.2 Gear: find necessary case hardness such that $(S_H)_G = (S_H)_p$
    - 4.2.1 Choose larger case hardness
    - 4.2.2 Check wear factor of safety S_H^2

*See **Example 14-8** from text*