15.1 Double integrals over rectangular regions.
15.2 Iterated integrals

Def^n: Let R be a region in $\mathbb{R}^2$, $f(x,y) \geq 0$ on R.

Then $\iint_R f(x,y)\, dA$ is the volume of the solid over R and below the surface $z = f(x,y)$.

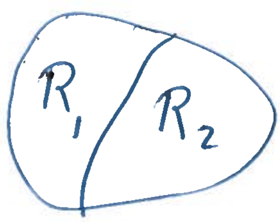
Th. 1) $\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

2) $\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$.

3) If $f(x,y) \geq g(x,y)$ on R, then

$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA.$$

4) $\iint_R f(x,y)\, dA =$

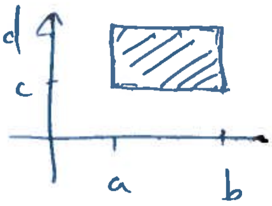


R

$$\iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA.$$

<u>Th.</u> let R be the rectangular region :

$b \geq x \geq a$, $d \geq y \geq c$, then



$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$$= \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx .$$

<u>Remark</u> : $\displaystyle\int_a^b \int_c^d f(x,y)\, dy\, dx =$

$$\int_c^d \int_a^b f(x,y)\, dx\, dy .$$

<u>Ex.</u> Find the volume of the solid over the rectangular region R ; $2 \geq y \geq 1$, $3 \geq x \geq 0$ and below the surface $z = 2x + 3y$.

<u>Soln.</u> Observe first that $z \geq 0$ over R .

Thus $\displaystyle V = \iint_R (2x+3y)\, dA =$

$$= \int_0^3 \int_1^2 (2x+3y)\,dy\,dx$$

$$= \int_0^3 \left[2xy + \frac{3y^2}{2}\right]_1^2 dx$$

$$= \int_0^3 \left[(4x+6) - \left(2x+\frac{3}{2}\right)\right] dx$$

$$= \int_0^3 \left(2x + \frac{9}{2}\right) dx =$$

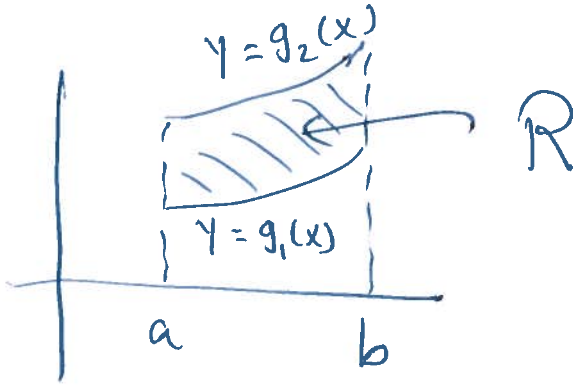
$$\left. x^2 + \frac{9}{2}x \right]_0^3 =$$

$$\left(9 + \frac{27}{2}\right) - (0+0) =$$

$$\frac{45}{2}$$

15.3 Double integrals over general regions.

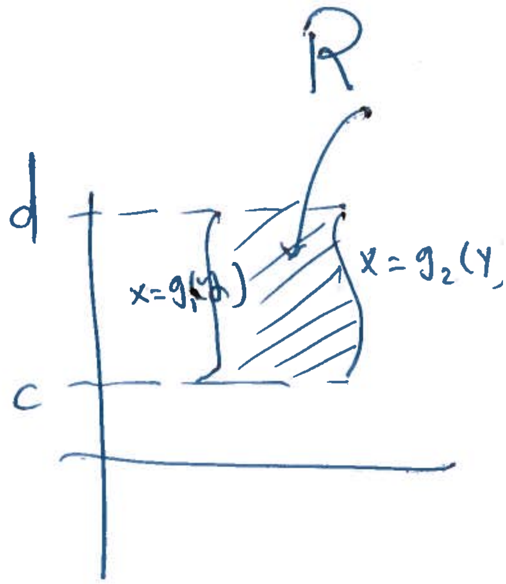
Th.



$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

$g_1(x) \le g_2(x)$ on $[a,b]$

Th.



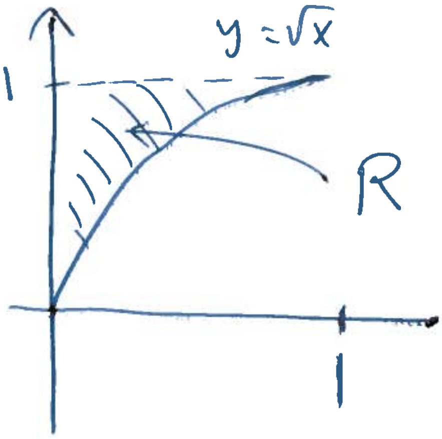
$$\iint_R f(x,y)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy.$$

$g_1(y) \le g_2(y)$ on $[c,d]$

Ex. Evaluate $\iint_R \sin y^3 \, dA$, where

R is the region enclosed by the curves:

$y = \sqrt{x}$, $x = 0$, $y = 1$



$\sqrt{x} = 1 \rightarrow$
$x = 1$

Soln.

$$\iint_R \sin y^3 \, dA =$$

$$\int_0^1 \int_0^{y^2} \sin y^3 \, dx \, dy =$$

$$\int_0^1 y^2 \sin y^3 \, dy =$$

$$\frac{1}{3} \int_0^1 3y^2 \sin y^3 \, dy = -\frac{1}{3} \cos y^3 \Big]_0^1$$

$$= -\frac{1}{3} (\cos 1 - \cos 0)$$

$$= -\frac{1}{3} (\cos 1 - 1).$$

Observe in the previous example that

$$\iint_R \sin y^3 \, dA = \int_0^1 \int_{\sqrt{x}}^1 \sin y^3 \, dy \, dx$$

~~that~~ cannot be calculated immediately. Therefore, we have reversed the order of integration to get easy integral calculations.

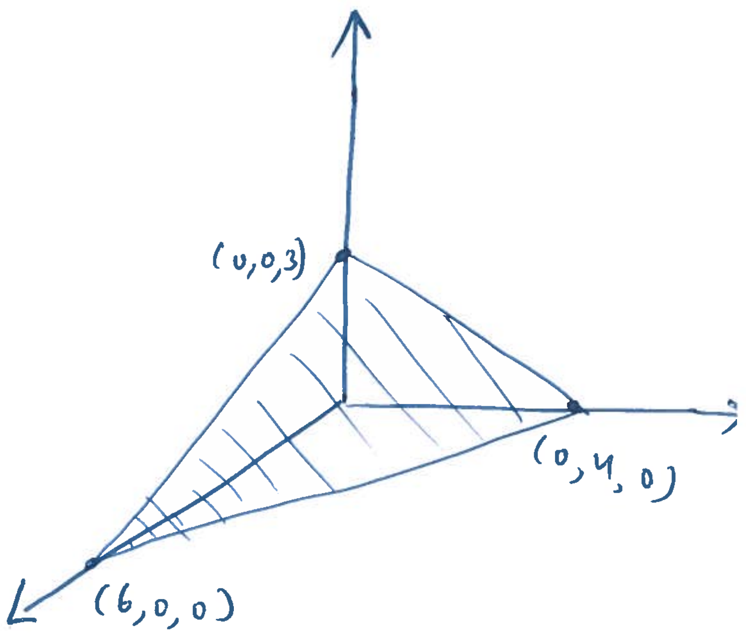
Ex. Find the volume of the solid enclosed by the surfaces: $2x + 3y + 4z = 12$, $x = 0$, $y = 0$, and $z = 0$.

Solution:

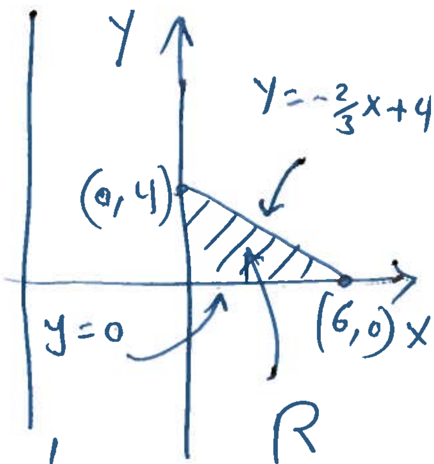
$x = 0, y = 0 \Rightarrow 0 + 0 + 4z = 12$
$\Rightarrow z = 3$

$x = 0, z = 0 \Rightarrow 0 + 3y = 12$
$\Rightarrow y = 4$

$y = 0, z = 0 \Rightarrow 2x + 0 = 12$
$\Rightarrow x = 6$



Observe that the solid is over the region R and below the surface $z = \frac{1}{4}(12-2x-3y)$, that is, $z \geq 0$ over R.



Thus $$V = \iint_R \frac{1}{4}(12-2x-3y)\,dA$$

$$= \int_0^6 \int_0^{-\frac{2}{3}x+4} \frac{1}{4}(12-2x-3y)\,dy\,dx$$

$$= \int_0^6 \frac{1}{4}\left(12y - 2xy - \frac{3y^2}{2}\right)\Big]_0^{-\frac{2}{3}x+4} dx$$

$$= \frac{1}{4}\int_0^6 \left[\left(12\left(-\frac{2}{3}x+4\right) - 2x\left(-\frac{2}{3}x+4\right) - \frac{3}{2}\left(-\frac{2}{3}x+4\right)^2\right]dx$$

$$\frac{y-0}{x-6} = \frac{4-0}{0-6}$$

$$\frac{y}{x-6} = -\frac{2}{3}$$

$$y = -\frac{2}{3}(x-6)$$

$$\boxed{y = -\frac{2}{3}x+4}$$

$$= \frac{1}{4}\int_0^6 \left[-8x+48+\frac{4}{3}x^2 - 8x - \frac{3}{2}\left(-\frac{2}{3}x+4\right)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \left[-16x+48+\frac{4}{3}x^2 - \frac{3}{2}\left(-\frac{2}{3}x+4\right)^2\right]dx$$

$$= \frac{1}{4}\left[-\frac{16x^2}{2} + 48x + \frac{4}{9}x^3 - \frac{3}{2}\left(-\frac{3}{2}\right)\frac{\left(-\frac{2}{3}x+4\right)^3}{3} \right]_0^6$$

$$= \frac{1}{4}\left(\left[\overset{-8(36)}{\cancel{-48}} + 48(6) + \frac{4}{9}(6)^3 + \frac{9}{4}\cancel{\frac{\left(-\frac{2}{3}(6)+4\right)^3}{3}}\right] - \left[0 + \frac{9}{4}\left(\frac{56}{3}\right)\right]\right)$$

$$= \frac{1}{4}\left(\left[\cancel{-288 + 288} + \frac{8}{3}(36)\right] - 48\right)$$

$$= \frac{1}{4}\left(\left[\cancel{16+}\ 96\right] - [48]\right)$$

$$\cancel{= \frac{1}{4}(96) = 28}$$

$$= \frac{1}{4}[48] = 12 .$$

Observe ; Immediately, you can get the volume as follows:

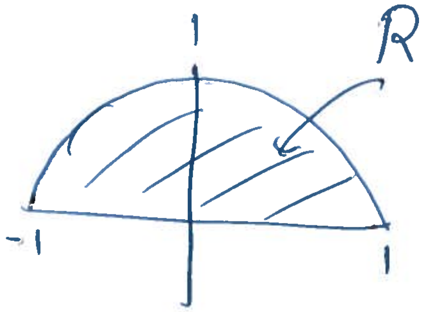
$$V = \frac{1}{3}(\text{area of the base})(\text{hight})$$

$$= \frac{1}{3}\left(\frac{1}{2}(6)(4)\right)(3)$$

$$= 12 .$$

Remark :

Ex. Evaluate $\iint x^2 y \, dA$, where R is the region enclosed by R the upper semi-circle $x^2+y^2=1$ and $y=0$.



$$\iint_R xy \, dA = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} x^2 y \, dy \, dx$$

$$= \int_{-1}^{1} \frac{x^2 y^2}{2} \Big]_{0}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left[\frac{x^2}{2}\left((1-x^2)-0\right) \right] dx .$$

$$= \int_{-1}^{1} \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx =$$

$$\frac{x^3}{6} - \frac{x^5}{10} \Big]_{-1}^{1} = \cancel{\left(\frac{1}{4} - \frac{1}{8}\right) - \left(\frac{1}{4} - \frac{1}{8}\right)}$$

$$= \left(\frac{1}{6} - \frac{1}{10}\right) - \left(-\frac{1}{6} + \frac{1}{10}\right) = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Exc. Evaluate $\iint_R x^2 y \, dA$, where

R is the closed circular region $x^2 + y^2 \le 1$

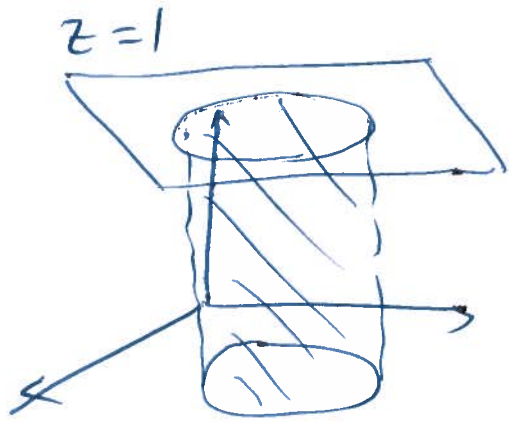
Exc. Evaluate $\iint_R x^2 y \, dA$, where

R is the region in the first quadrant

enclosed by the curves:

$x^2 + y^2 = 1, \; x = 0, \; y = 0.$

Remark:



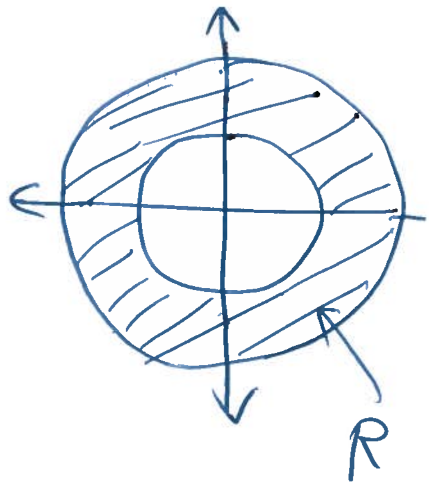
$$\iint_R dA = \text{the area of } R = A(R)$$

because:

$$\iint_R 1 \, dA = \text{volume of the solid with base } R \text{ and height } 1$$
$$= [A(R)][1]$$
$$= A(R).$$

Ex. Let R be the region enclosed by the curves $x^2+y^2=1$ & $x^2+y^2=4$.

Evaluate $\iint_R 4\, dA$.



Solution:

$$\iint_R 4\, dA = 4 \iint_R dA$$

$$= 4\ A(R)$$

$$= 4\left[(2)^2\pi - (1)^2\pi\right]$$

$$= 4\,[3\pi]$$

$$= 12\pi.$$

Ex: Estimate $\iint_R e^{\sin x}\, dA$, where R is the closed disc $x^2+y^2 \le 9$.

Soln: $1 \ge \sin x \ge -1$

$$e \ge e^{\sin x} \ge e^{-1} \quad \text{on } R.$$

$$\therefore \iint_R e\, dA \ge \iint_R e^{\sin x}\, dA \ge \iint_R e^{-1}\, dA$$

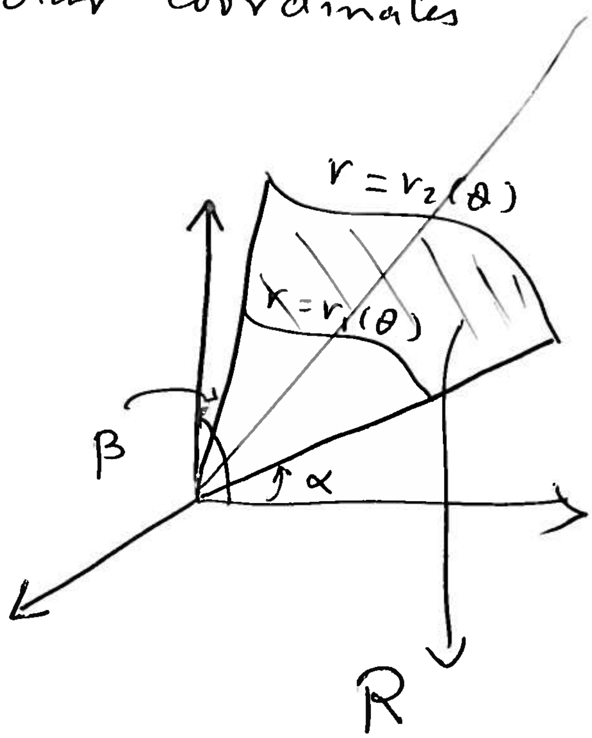
$$e\, A(R) \ge \iint_R e^{\sin x}\, dA \ge e^{-1} A(R)$$

$$e\left((3)^2 \pi\right) \ge \iint_R e^{\sin x}\, dA \ge e^{-1}\left((3)^2\pi\right)$$

$$9\pi e \ge \iint_R e^{\sin x}\, dA \ge \frac{9\pi}{e}.$$

15.4 Double integrals in polar coordinates

Th. $$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} r\, f(r,\theta)\,dr\,d\theta.$$



R : simple polar region

~~$r_1(\theta) \geq 0$~~

~~$r_2(\theta) \geq 0$~~

$0 \leq r_1(\theta) \leq r_2(\theta) \quad \forall \theta \in [\alpha, \beta]$

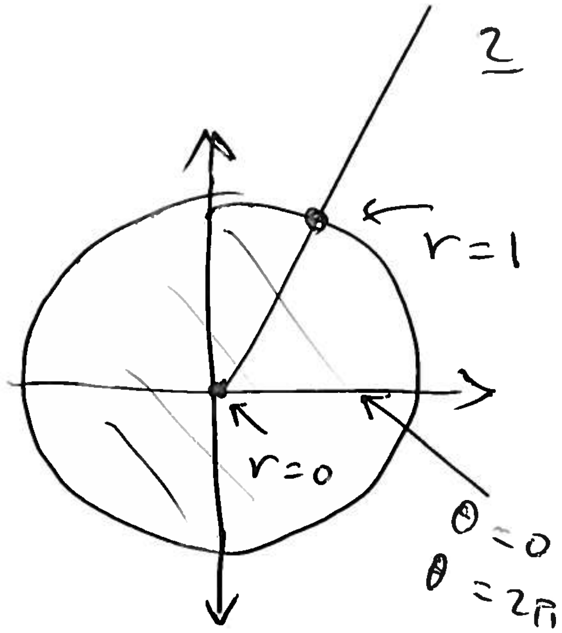
$\alpha \leq \beta$

$0 \leq \beta - \alpha \leq 2\pi$

Ex. Evaluate $\iint_R e^{x^2+y^2}\, dA$

where R is the circular region

$x^2 + y^2 \leq 1$



$$\iint_R e^{x^2+y^2}\,dA =$$

$$\int_0^{2\pi}\int_0^1 r e^{r^2}\,dr\,d\theta =$$

$$\int_0^{2\pi} \frac{1}{2}\int_0^1 2r e^{r^2}\,dr\,d\theta =$$

$$\frac{1}{2}\int_0^{2\pi} e^{r^2}\Big]_0^1\,d\theta = \frac{1}{2}\int_0^{2\pi}(e-1)\,d\theta$$

$$= \frac{1}{2}(e-1)\,2\pi$$

$$= \pi(e-1).$$

Ex: Evaluate $\iint_R e^{x^2+y^2}\,dA$,

where R is the region enclosed between

$x^2+y^2=1$ & $x^2+y^2=4$

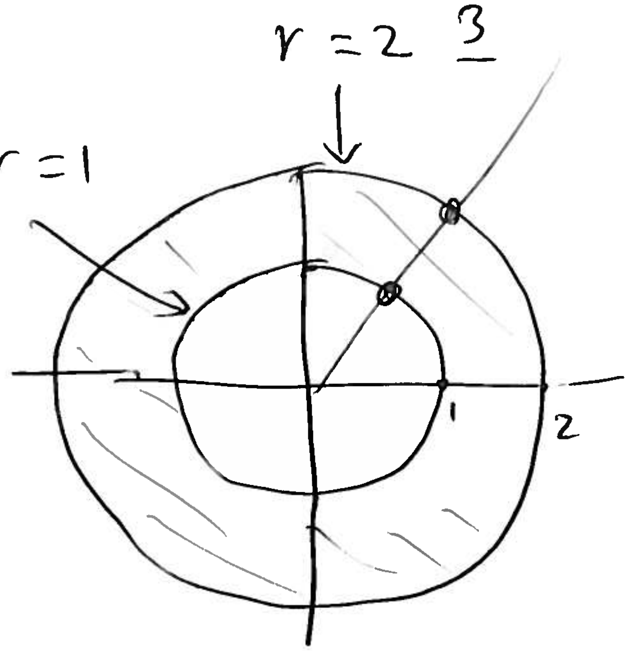
$$\int_0^{2\pi}\int_1^2 r e^{r^2}\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} e^{r^2}\Big]_1^2\,d\theta =$$

$$\frac{1}{2}\int_0^{2\pi}(e^4 - e)\,d\theta =$$

$$\frac{1}{2}(2\pi)(e^4 - e) =$$

$$\pi(e^4 - e).$$



$x^2 + y^2 = 4$

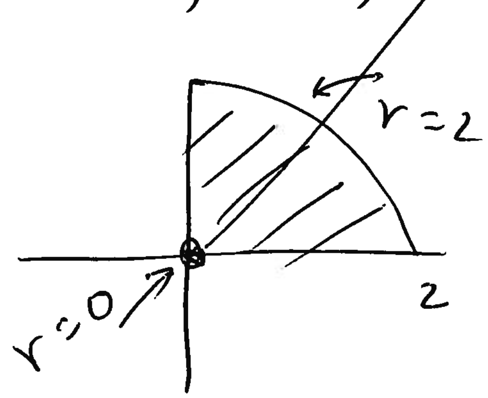
$r^2 = 4$

$r = 2$

Ex: Evaluate $\iint_R e^{x^2+y^2}\,dA$,

where R is the region in the first quadrant enclosed by $x^2+y^2=4$, $x=0$, $y=$

Soln



$\underline{u}$

$$\iint_R e^{x^2+y^2}\, dA =$$

$$\int_0^{\pi/2}\int_0^1 r e^{r^2}\, dr\, d\theta =$$

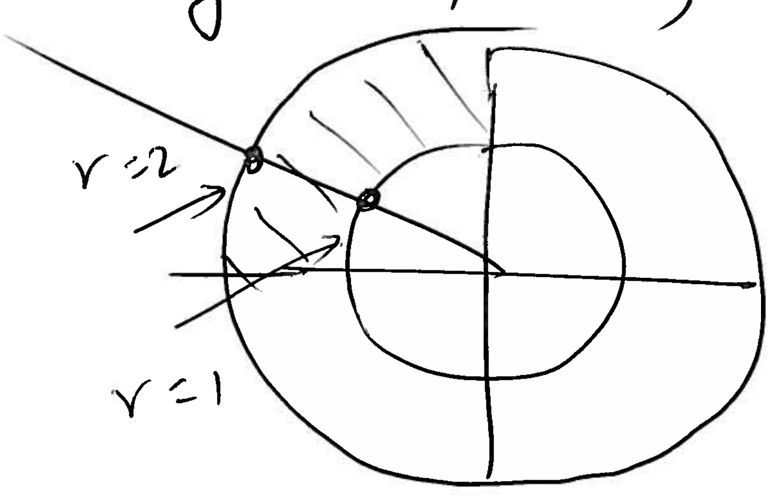
$$\frac{1}{2}\int_0^{\pi/2} e^{r^2}\Big]_0^1 \, d\theta =$$

$$\frac{1}{2}\int_0^{\pi/2} (e-1)\, d\theta =$$

$$\frac{1}{2}\left(\frac{\pi}{2}\right)(e-1) = \frac{\pi}{4}(e-1).$$

Ex: Evaluate $\displaystyle\iint_R \frac{1}{1+x^2+y^2}\, dA$,

where R is the region in the second quadrant enclosed by $x^2+y^2=1$, $x^2+y^2=4$.



$\underline{5}$

$$\int_{\frac{\pi}{2}}^{\pi} \int_{1}^{2} \frac{r}{1+r^2} \, dr \, d\theta =$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} \ln(1+r^2) \Big]_{1}^{2} \, d\theta =$$

$$\frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (\ln 5) - \ln 2) \, d\theta =$$

$$\frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \ln\left(\frac{5}{2}\right) d\theta$$

$$= \frac{1}{2} \ln\left(\frac{5}{2}\right)\left(\pi - \frac{\pi}{2}\right)$$

$$= \frac{\pi}{4} \ln\left(\frac{5}{2}\right).$$

15.7 Tripple integrals

<u>Defn</u>: $\iiint_G dV$ is the ~~sol~~. volume of the solid G.

<u>Th.</u> $\iiint_G f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$,

where G is the rectangular box:

$a \le x \le b$, $c \le y \le d$, $e \le z \le f$.

<u>Remark</u>: In the above interated triple integral, changing the order of integration gives the same value, e.g.

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

<u>Ex.</u> let G be the box: $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$. Find $\iiint_G xyz\,dV$.

$$\int_0^3\int_0^2\int_0^1 xyz\,dx\,dy\,dz =$$

$$\int_0^3\int_0^2 \frac{x^2}{2}yz\Big]_0^1\,dy\,dz =$$

$$\int_0^3\int_0^2 \frac{1}{2}yz\,dy\,dz =$$

$$\int_0^3 \frac{y^2}{4}z\Big]_0^2\,dz =$$

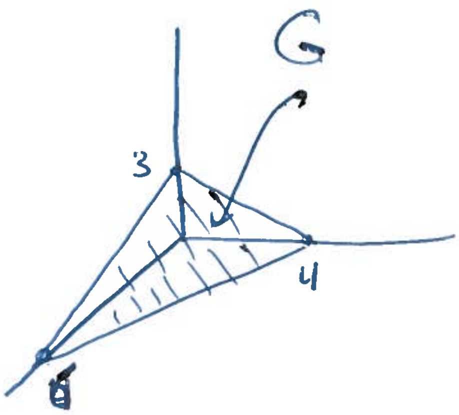
$$\int_0^3 z\,dz = \frac{z^2}{2}\Big]_0^3 = \frac{9}{2}.$$

~~Ex.~~ Th. let G be a solid in $\mathbb{R}^3$ bounded below by the surface $z = g_1(x,y)$ and above by the surface $z = g_2(x,y)$. If R is the projection of the solid on the xy-plane, then:

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Ex. Find the volume of the solid enclosed by the surface $2x+3y+4z=12$, $x=0$, $y=0$, $z=0$

Soln.



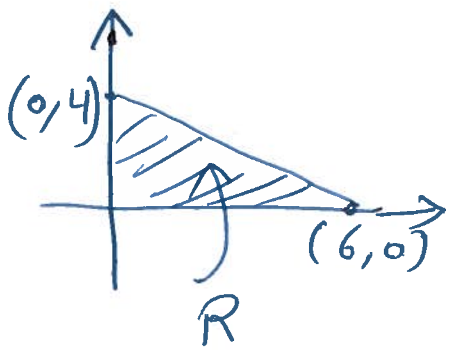
$$V_G = \iiint_G dV$$

$$\cancel{\int_{0}^{\frac{1}{4}(12-2x-3y)}}$$

$z = 0$ (lower surface)

$z = \frac{1}{4}(12-2x-3y)$ (upper surface)

$$= \iint_R \left[\int_0^{\frac{1}{4}(12-2x-3y)} dz \right] dA$$



$$= \iint_R \frac{1}{4}(12-2x-3y)\, dA$$

R is the projection of G on the xy-plane

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} \frac{1}{4}(12-2x-3y)\, dy\, dx$$

$$\frac{y-0}{x-6} = \frac{4-0}{0-6}$$

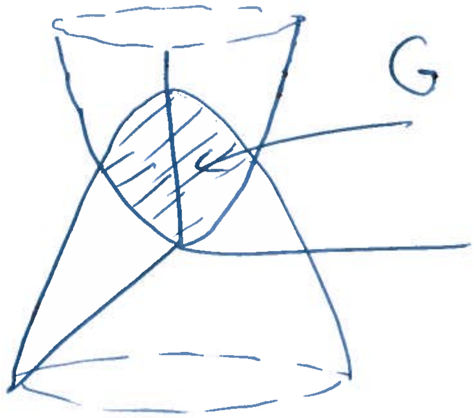
$$y = -\frac{2}{3}(x-6)$$

$$y = 4 - \frac{2}{3}x$$

$= 12$ ~~(check!)~~.

(calculations are done before).

Ex. Using rectangular coordinates, express the volume of the solid enclosed by the surfaces $z = x^2 + y^2$, $z = 8 - x^2 - y^2$ as an iterated triple integral (Do not evaluate).



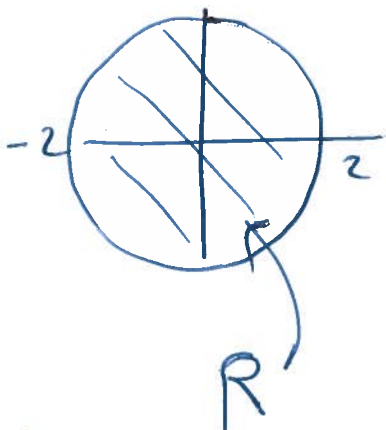
Soln. $x^2 + y^2 = 8 - x^2 - y^2$

$2x^2 + 2y^2 = 8$

$x^2 + y^2 = 4$

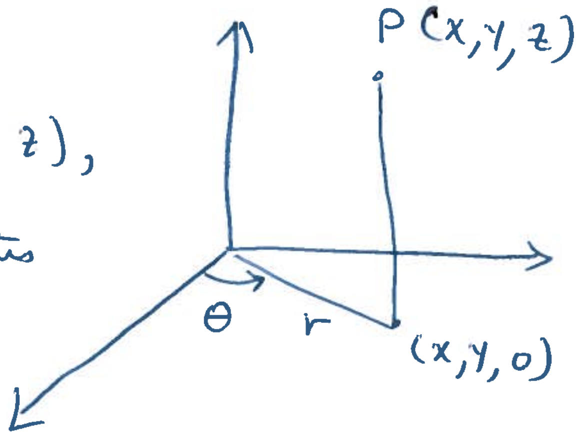
$$V_G = \iiint_G dV$$

$$= \iint_R \left[\int_{x^2+y^2}^{8-x^2-y^2} dz \right] dA$$



$$= \cancel{\iint_R} \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\, dy\, dx .$$

15.8 Triple integrals in cylindrical coordinates.



Def$^{\underline{n}}$: If the rectangular coordinates of a point P are (x, y, z), then the ~~xx~~ cylindrical coordinates of P are (r, θ, z)

We will usually restrict θ as
$2\pi > \theta \geq 0$

$$r = \sqrt{x^2+y^2}$$
$$\tan\theta = \frac{y}{x}.$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$2\pi > \theta \geq 0$$

Ex. Find cylindrical ~~points~~ coordinates for the point with rectangular coordinates $\overset{(x,y,z)}{(-2, 2, 3)}$.

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}.$$

$$\tan\theta = \frac{y}{x} = \frac{2}{-2} = -1$$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ (we took the $2^{\underline{d}}$ quadrant because $x<0, y>0$)

$\therefore$ cylind. coordinates are $\left(2\sqrt{2}, \frac{3\pi}{4}, 3\right)$.

Ex: Find cylind. coord. of the point with rectangular coordinates $\underset{(x,\ y,\ z)}{(3, -\sqrt{3}, -2)}$.

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}.$$

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}.$$

$\theta = \cancel{\frac{\pi}{6} + \pi = \frac{7\pi}{6}}$ (we took the ~~third~~ fourth quadrant because $x > 0,\ y < 0$).
$= 2\pi - \frac{\pi}{6} = 11\pi/6$

$\therefore$ the cylindrical coord. are $(2\sqrt{3}, \frac{11\pi}{6}, -2)$.

Ex: Find rectangular coordinates of the point with cylindrical coordinates ~~$(-2, 2, 1)$~~.
$(4, \frac{4\pi}{3}, 1)$.
(r, θ, z)

$$x = r\cos\theta = 4\cos\frac{4\pi}{3}$$
$$= 4\left(-\frac{1}{2}\right) = -2$$
$$y = r\sin\theta = 4\sin\frac{4\pi}{3} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}.$$

$\therefore$ ~~cylindrical~~ rectangular coordinates are $(-2, -2\sqrt{3}, 1)$

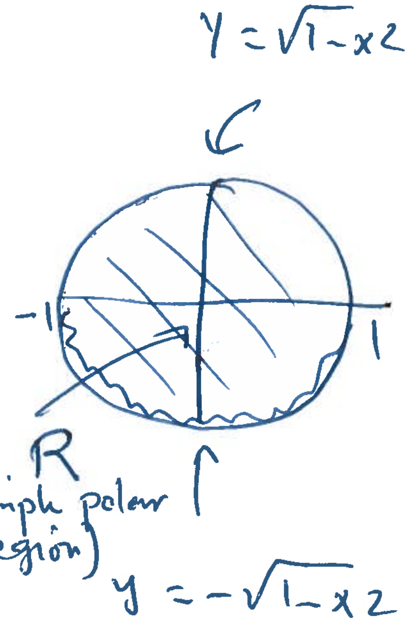
Ex. The following iterated triple integral is given in rectangular coordinates. Express it an an iterated triple integral using cylindrical coordinates, then evaluate it

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^2 \, dz\, dy\, dx.$$

Cobserve that calculations are not straightforward.

Soln : $z = x^2+y^2$ (lower surface)

$z = 2 - x^2 - y^2$ (upper surface).

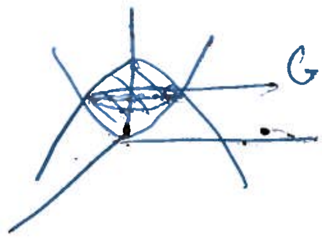


R is the projection of the solid bounded below by $z = x^2+y^2$ and above by $z = 2-x^2-y^2$.

So

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^2 \, dz\, dy\, dx =$$

$2-x^2-y^2 = x^2+y^2 \Rightarrow$
$2x^2+2y^2 = 2 \Rightarrow$
$\boxed{x^2+y^2=1}$



$$\iint_R \left[\int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^2 \, dz \right] dA =$$

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r\,(r^2)^2 \, dz\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r^5 \, dz\, dr\, d\theta =$$

$$\cancel{\int_0^{2\pi} \int_0^1 \left[\frac{r^6}{6} \right]_{r^2}^{2-r^2}}$$

$$\int_0^{2\pi} \int_0^1 r^5 (2-r^2-r^2)\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 r^5 (2-2r^2)\, dr\, d\theta =$$

$$2\int_0^{2\pi} \int_0^1 r^5 (1-r^2)\, dr\, d\theta =$$

$$2\int_0^{2\pi} \int_0^1 (r^5 - r^7)\, dr\, d\theta =$$

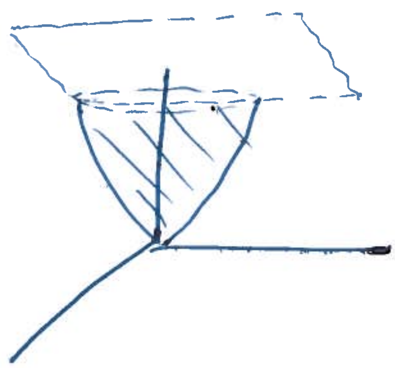
$$2\int_0^{2\pi} \frac{r^6}{6} - \frac{r^8}{8} \Big]_0^1 \, d\theta =$$

$$2\int_0^{2\pi} \left(\frac{1}{6} - \frac{1}{8}\right) d\theta =$$

$$2\int_0^{2\pi} \frac{1}{24} \, d\theta = 2(2\pi)\frac{1}{24} = \frac{1}{6}\pi$$

(observe that calculations became easy after transforming into cylindrical coordinates).

Ex. Find the volume of the solid inside the elliptic paraboloid $z = x^2 + y^2$ and below the plane $z = 9$.

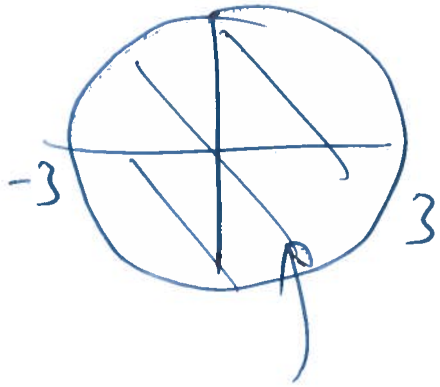


Soln.

$\boxed{x^2 + y^2 = 9}$ is the ~~projection~~ circle of intersection

$$V_G = \iiint_G dV$$

$$= \iint_R \left[\int_{x^2+y^2}^{9} dz \right] dA$$



R

$$= \int_0^{2\pi} \int_0^{3} \int_{r^2}^{9} \boxed{r}\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{3} r(9-r^2)\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^{3} (9r - r^3)\, dr\, d\theta =$$

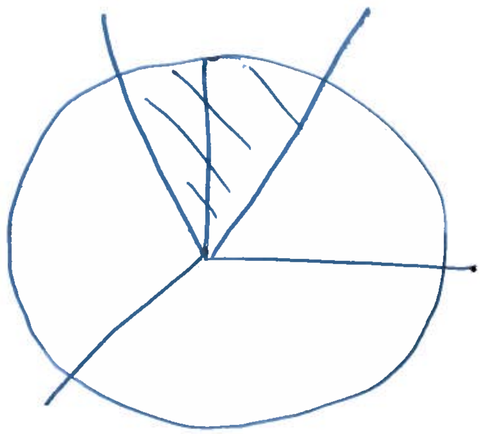
$$\int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3 d\theta =$$

$$\int_0^{2\pi} \left(\frac{81}{2} - \frac{81}{4} \right) d\theta = \int_0^{2\pi} \frac{81}{4}\, d\theta$$

$$= \frac{81}{4}(2\pi) = \frac{81}{2}\pi.$$

Ex. Find the volume of the solid inside the upper cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 4$

Soln,

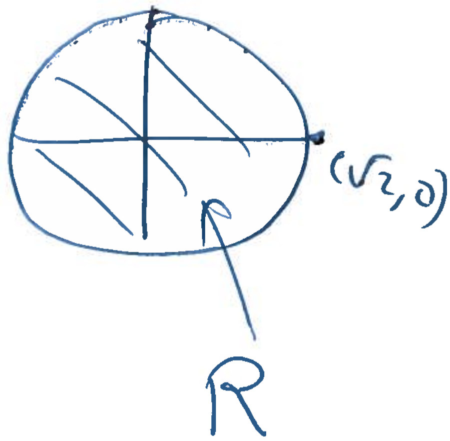


$$x^2+y^2+x^2+y^2 = 4$$

$$x^2+y^2 = 2$$

$$V = \iiint_G dV$$

$$= \iint_R \left[\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} dz \right] dA$$



$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r(\sqrt{4-r^2} - r) \, dr \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (r\sqrt{4-r^2} - r^2) \, dr \, d\theta =$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} \frac{(u-r^2)^{3/2}}{3/2} - \frac{r^3}{3} \Big]_0^{\sqrt{2}} \right) d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{3} \sqrt{(u-r^2)^3} - \frac{r^3}{3} \right) \Big]_0^{\sqrt{2}} d\theta$$

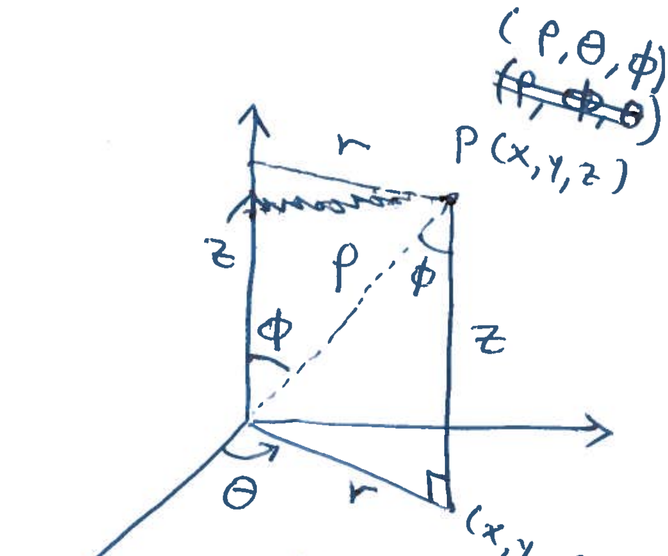
$$= \int_0^{2\pi} \left[\left(\cancel{-\frac{1}{3}} (\sqrt{8} - \frac{2\sqrt{2}}{3} \right) - \left(-\frac{8}{3} - 0 \right) \right] d\theta$$

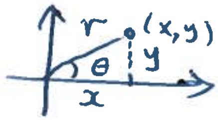
$$= \int_0^{2\pi} \frac{8 - 2\sqrt{2} - \sqrt{8}}{3} d\theta = \int_0^{2\pi} \frac{8 - 2\sqrt{8}}{3} d\theta$$

$$2\pi \frac{(8 - 2\sqrt{8})}{3} =$$

$$\frac{2\pi}{3} (8 - 4\sqrt{2}).$$

15.9 Triple integrals in spherical coordinates.





Def^n: If the rectangular coordinates of a point P are (x, y, z), then the spherical coordinates of P are (ρ, θ, ϕ), where $0 \leq \phi \leq \pi$ is the angle between the positive z-axis and the segment joining P and the origin.

Restric also θ as $2\pi \geq \theta \geq 0$

$z = \rho \cos\phi$

$r = \rho \sin\phi$

$\tan\phi = \frac{r}{z}$

$\rho = \sqrt{x^2 + y^2 + z^2}$

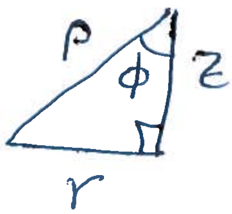
$x = r \cos\theta$

$x = \rho \sin\phi \cos\theta$

$y = r \sin\theta$

$y = \rho \sin\phi \sin\theta$

$\rho = \sqrt{r^2 + z^2}$



Ex. The point $\left(2, \frac{5\pi}{4}, \frac{2\pi}{3}\right)$ is given in spherical coordinates. Find the rectangular coordinates of the point.

$$(\rho, \theta, \phi)$$
$$\left(2, \frac{5\pi}{4}, \frac{2\pi}{3}\right)$$

$$x = \rho \sin\phi \cos\theta$$
$$= 2 \sin\frac{2\pi}{3} \cos\frac{5\pi}{4}$$
$$= (2)\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) = -\sqrt{\frac{3}{2}}.$$

$$y = \rho \sin\phi \sin\theta$$
$$= 2 \sin\frac{2\pi}{3} \sin\frac{5\pi}{4}$$
$$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{\frac{3}{2}}.$$

$$z = \rho \cos\phi$$
$$= 2 \cos\frac{2\pi}{3}$$
$$= 2\left(-\frac{1}{2}\right) = -1$$

$\therefore$ spherical coordinates of the point are

$$\left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}, -1\right).$$

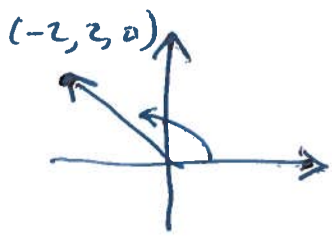
Ex. The point $(-2, 2, -\sqrt{8})$ is given in rectangular coordinates. Find the spherical coordinates of the point.

(x, y, z)
$(-2, 2, -\sqrt{8})$

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}.$$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+4+\cancel{16}\,8} \cancel{= \sqrt{24} = 2\sqrt{6}}.$$
$$= 4$$

$$\tan\theta = \frac{y}{x}$$
$$= \frac{2}{-2} = -1 \implies \theta = \pi - \frac{\pi}{4}$$
$$= \frac{3\pi}{4}.$$



(observe that $x<0$, $y>0$)

$$\tan\phi = \frac{r}{z} = \frac{\sqrt{8}}{-\sqrt{8}} \cancel{= \frac{2\sqrt{2}}{-4} = -\frac{\sqrt{2}}{2} \quad \phi = \ldots}.$$
$$= -1$$

$\therefore \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. (You can't choose ϕ in the fourth quadrant as $2\pi \geq \phi \geq 0$).

(to find ϕ, you can use also $r = \rho\sin\phi$ or

$$\boxed{z = \rho\cos\phi}$$

$$-\sqrt{8} = \cancel{2\sqrt{6}}\,4\cos\phi \implies \cancel{\cos\phi = -\frac{4}{2\sqrt{6}} = -\frac{2}{\sqrt{6}} = -\frac{2}{\sqrt{2}\sqrt{3}}}$$

$$\therefore \cos\phi = \frac{-\sqrt{8}}{4} = \frac{-2\sqrt{2}}{4}$$
$$= -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$$

$$\implies \phi = \pi - \frac{\pi}{4} = 3\pi/4 \text{)}$$

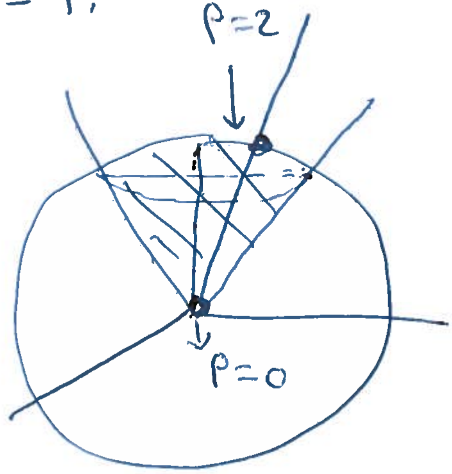
$\therefore$ spherical coordinates of the point are:

$$(\rho, \theta, \phi) = \left(4, \frac{3\pi}{4}, \frac{3\pi}{4}\right).$$

Th. $\iiint\limits_G f(x,y,z)\, dV =$

$$\underset{\substack{\text{appropiate}\\ \text{limits}}}{\iiint} \rho^2 \sin\phi\, f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, d\rho\, d\phi\, d\theta$$

Ex. Use spherical coordinates to find the volume of the solid inside the upper cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 4$.



Soln $z = r$

$\tan\phi = \frac{r}{z} = 1 \rightarrow$

$\phi = \pi/4$.

$$V_G = \iiint\limits_G dV$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3} \sin\phi \Big]_0^2 \, d\phi \, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{8}{3} \sin\phi \, d\phi \, d\theta =$$

$$= \int_0^{2\pi} -\frac{8}{3} \cos\phi \Big]_0^{\pi/4} \, d\theta$$

$$= \int_0^{2\pi} -\frac{8}{3} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta$$

$$= -\frac{8}{3} (2\pi) \left(\frac{1}{\sqrt{2}} - 1 \right).$$

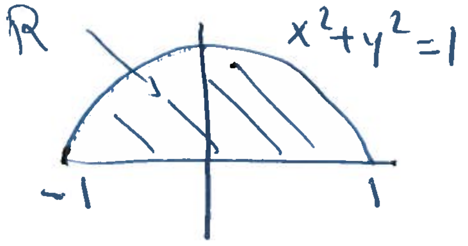
Ex. Express the interated triple integral

$$\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} \, dz \, dy \, dx$$

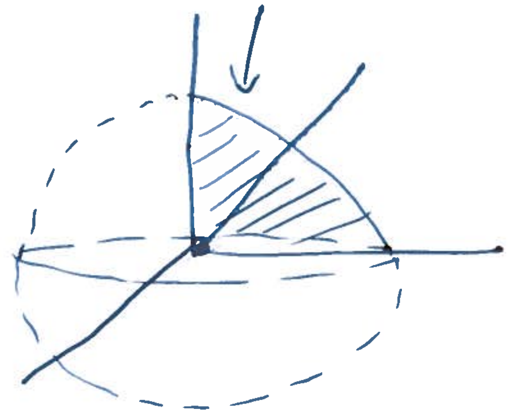
using spherical coordinates, then evaluate it.

First of all, we determine the solid

The projection is R



$(y>0)$



$\therefore \quad \pi \geq \theta \geq 0$

$$\int_0^{\pi}\int_0^{\pi/2}\int_0^{1} \rho^2 \sin\phi \left(e^{\rho^2}\right)^{3/2} d\rho\, d\phi\, d\theta$$

$z = \sqrt{1-x^2-y^2}$

$x^2+y^2+z^2=1$

$\rho^2=1$

$\rho=1$

$$\int_0^{\pi}\int_0^{\pi/2}\int_0^{1} \rho^2 e^{\rho^3} \sin\phi\, d\rho\, d\phi\, d\theta =$$

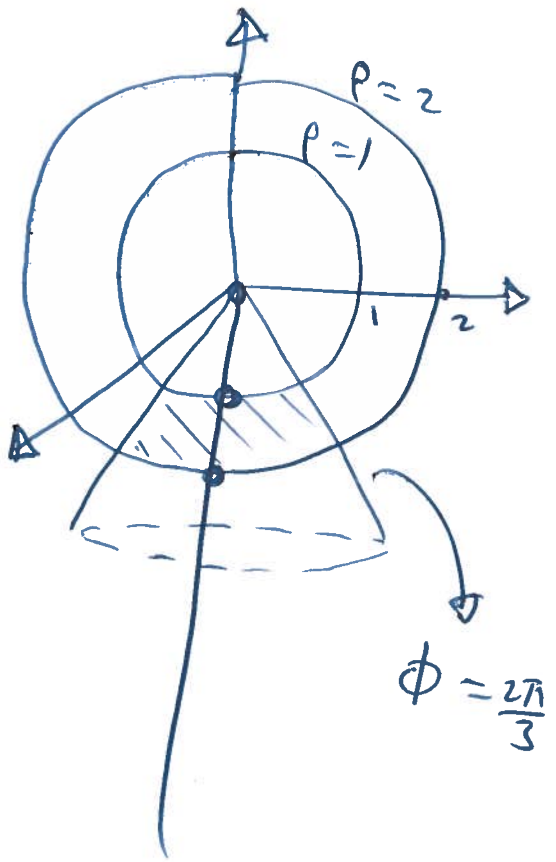
$$\frac{1}{3}\int_0^{\pi}\int_0^{\pi/2} e^{\rho^3} \sin\phi \Big]_0^1 d\phi\, d\theta =$$

$$\frac{1}{3}\int_0^{\pi}\int_0^{\pi/2} \sin\phi\,(e-1)\, d\phi\, d\theta =$$

$$= -\frac{1}{3}\int_0^{\pi} (e-1)\cos\phi \Big]_0^{\pi/2} d\theta =$$

$$= -\frac{1}{3}\int_0^{\pi} (e-1)(0-1)\, d\theta = -\frac{1}{3}(\pi)(1-e) = \frac{\pi}{3}(e-1).$$

Ex. Find the volume of the solid enclosed by the surfaces $\rho = 1$, $\rho = 2$ and inside the cone $\phi = \frac{2\pi}{3}$



Soln

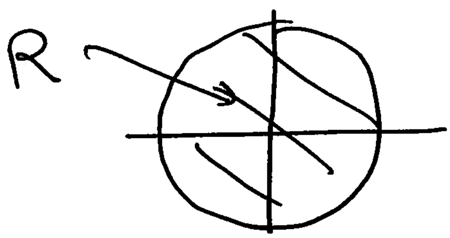
$$V_G = \iiint_G dV$$

$$= \int_0^{2\pi} \int_{\pi}^{2\pi/3} \int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_{\pi}^{2\pi/3} \frac{7}{3} \sin\phi \, d\phi \, d\theta$$

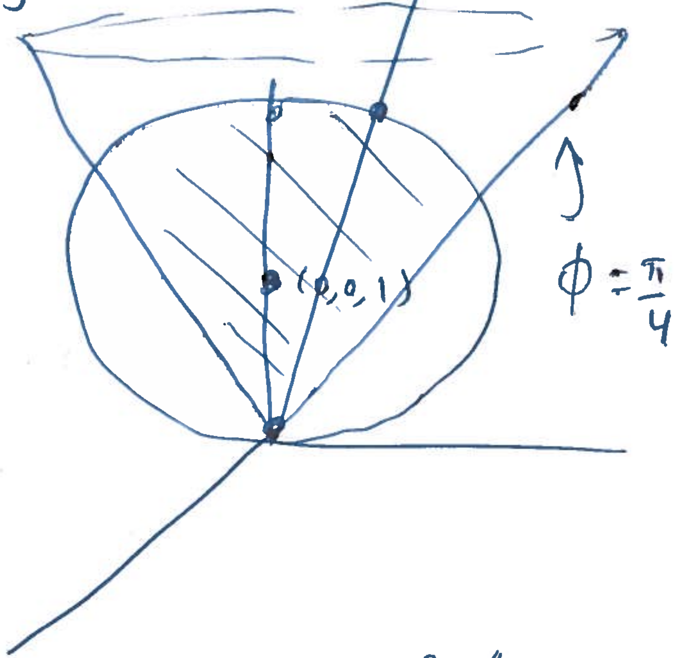
$$= \int_0^{2\pi} -\frac{7}{3} \left(\cos\frac{2\pi}{3} - \cos\pi \right) d\theta$$

$$= \int_0^{2\pi} -\frac{7}{3} \left(+\frac{1}{2} \right) d\theta = \frac{7}{6}(2\pi) = \frac{7\pi}{3}.$$



$2\pi \geq \theta \geq 0$

Ex. Express ~~Find~~ the volume of the solid ~~of the solid~~ above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=2z$ as an iterated triple integral using spherical coordinates (do not evaluate).



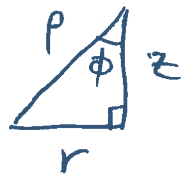
Soln. $x^2+y^2+(z-1)^2=1$

$\rho^2=2z$

but $z=\rho\cos\phi$.

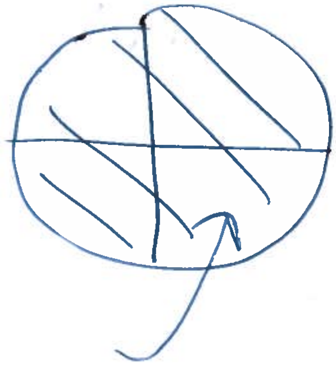
$\rho^2=2\rho\cos\phi$

$\rho=2\cos\phi$.



$$V_G=\iiint_G dV$$

$$=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$



R : projection

$2\pi\geq\theta\geq0$