14.1 Functions of several variables

Def^n: A fn. f ~~(x,y)~~ of two variables is a rule that associates to (x,y) a unique value $f(x,y)$

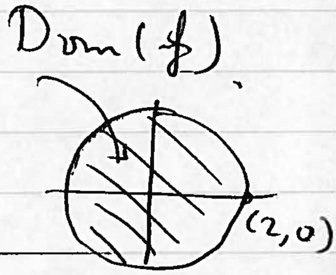
$\mathrm{Dom}(f)$ = set of (x,y)

$\mathrm{Rng}(f)$ = set of $f(x,y)$

$\mathrm{Graph}(f)$ = set of $(x,y,f(x,y))$.

~~Ex.~~ The level curves of f are $f(x,y)=K$, K constant

Ex. $f(x,y)=\sqrt{4-x^2-y^2}$



$4-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 4.$

$\mathrm{Dom}(f)=\{(x,y): x^2+y^2 \leq 4\}$

= the closed disc with center $(0,0)$ and radius 2.

To find $\mathrm{Rng}(f)$:

let $(x,y) \in \mathrm{Dom}(f)$

$0 \leq x^2+y^2 \leq 4$ $\qquad\qquad 0 \geq -x^2-y^2 \geq -4$

$4 \geq 4-x^2-y^2 \geq 0$

$2 \geq z=\sqrt{4-x^2-y^2} \geq 0$

$\therefore \mathrm{Rng}(f)=[0,2]$.

To identify and sketch the graph of f:

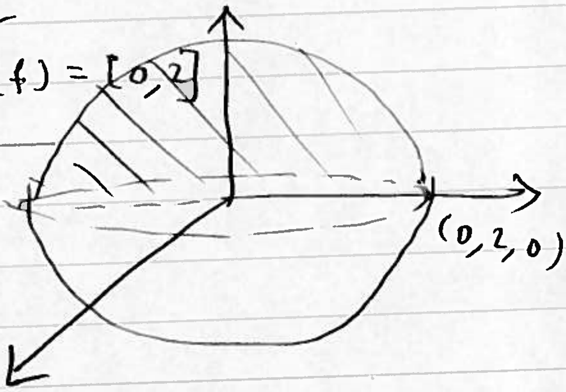
$z=\sqrt{4-x^2-y^2}$

$x^2+y^2+z^2=4$ (sphere).

but $z \geq 0$, so

Graph(f) is the upper hemisphere with center (0,0,0) and radius 2

[from the graph, you can see that Rng(f) = [0,2]]

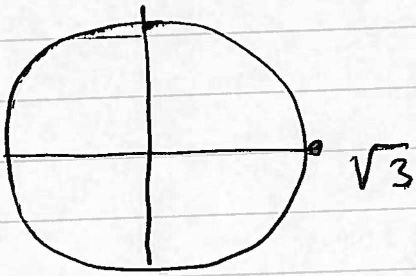


- Find the level curve of f for k = 1 :

$f(x,y)=1$

$\sqrt{4-x^2-y^2}=1$

$x^2+y^2=3$



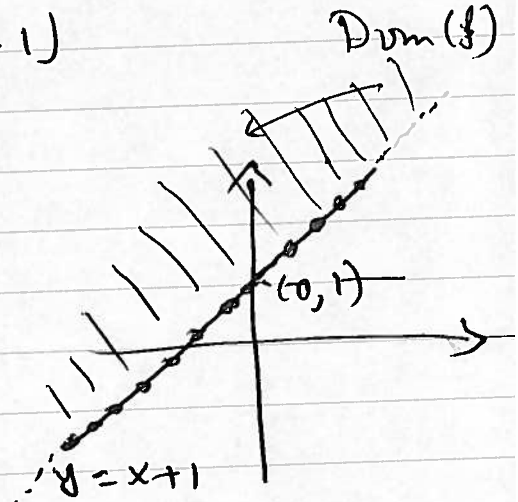
Ex. $f(x,y)=\ln(y-x-1)$

$y-x-1>0$

$y>x+1$

$Dom(f)=\{(x,y) : y>x+1\}$

$Rng(f)=(\cancel{0}\,{-\infty},\infty)=\mathbb{R}$

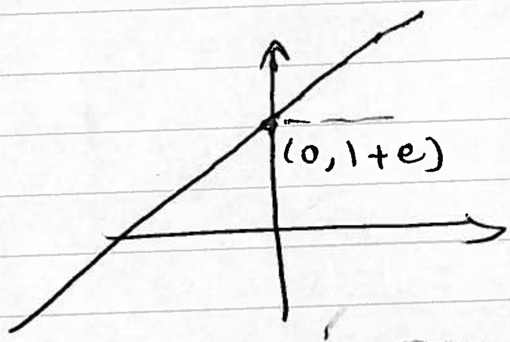


level curve of f for k = 1 is

$f(x,y)=1$

$\ln(y-x-1)=1$

$y-x-1=e \longrightarrow y=x+1+e$



Ex. Identify & sketch the graph of

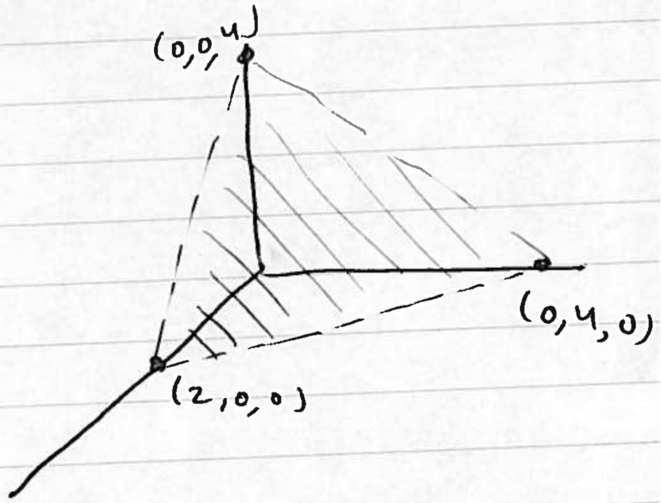
$f(x,y) = 4 - 2x - y$

$z = 4 - 2x - y$ (plane).

z-intercept : $\cancel{x} = 0$ & $y = 0 \to z = 4$

x-intercept : $y = 0$ & $z = 0 \to x = 2$

y-intercept : $x = 0$ & $z = 0 \to y = 4$



$f(x,y) = 4 - 2x - y$

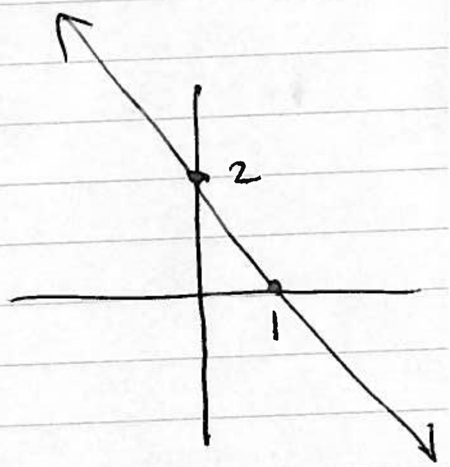
$f(1,2) = 4 - 2 - 2 = 0$.

level curve of f for $k = 2$ is:

$f(x,y) = 2$

$4 - 2x - y = 2 \to \cancel{2x}$

$y = 2 - 2x$



Def^n: The level surface of $f(x,y,z)$ is $f(x,y,z)=k$.

Ex. $f(x,y,z) = x^2+y^2+z^2$.

Dom$(f) = \mathbb{R}^3$.

Rng$(f) = [0,\infty)$.

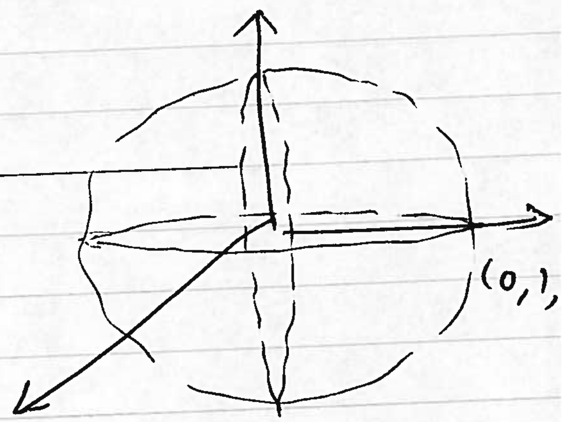
$f(1,2,-3) = 1+4+9 = 14$.

level surface of f for $k=1$ is:

$f(x,y,z)=1$

$x^2+y^2+z^2=1$

(sphere with center $(0,0,0)$ and radius 1).



14.2 Limits and continuity

Ex. $$\lim_{(x,y)\to(0,0)} \frac{\sin(2x^2+2y^2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin 2(x^2+y^2)}{x^2+y^2} \qquad \text{let } t = x^2+y^2$$

$$= \lim_{t\to 0^+} \frac{\sin 2t}{t} = 2.$$

Ex. $$\lim_{(x,y,z)\to(0,0,0)} e^{-1/x^2+y^2+z^2} \qquad \text{let } t = x^2+y^2+z^2$$

$$= \lim_{t\to 0^+} e^{-\frac{1}{t}} = e^{-\infty} = 0.$$

Ex. $$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2-y^2} =$$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{(x^2-y^2)} = \lim_{(x,y)\to(0,0)} x^2+y^2 = 0+0=0$$

Ex. $$\lim_{(x,y,z)\to(0,0,0)} \frac{e^{x^2+y^2+z^2}}{x^2+y^2+z^2} \qquad \text{let } t = x^2+y^2+z^2$$

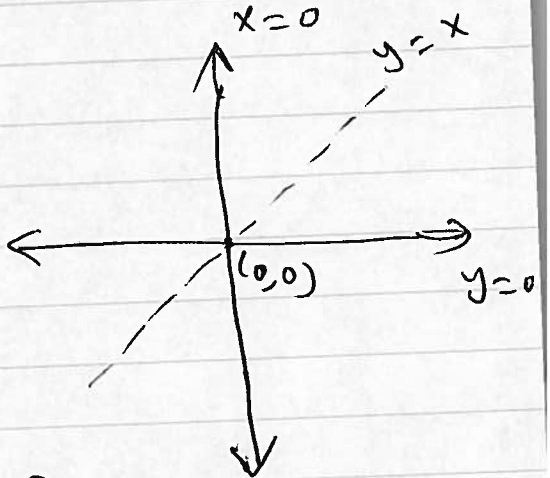
$$= \lim_{t\to 0^+} \frac{e^t}{t} = +\infty \quad (\nexists) \quad \cancel{= \lim_{t\to 0^+} \frac{e^t}{t} = e^0 = 1.}$$

polynomial in two variables

Ex. $$\lim_{(x,y)\to(2,1)} \frac{x^2+3y^2}{x^2-y^2} = \frac{4+3}{4-1} = \frac{7}{3}.$$

Ex. $$\lim_{(x,y)\to(0,0)} \frac{2xy\cancel{+3y^2}}{3x^2+y^2}$$

rational fn. in two variables



(i) along $y=0$:

$$\lim_{x\to 0} \frac{2x(0)}{3x^2+0} = \lim_{x\to 0} 0 = 0$$

(ii) along $x=0$:

$$\lim_{y\to 0} \frac{2(0)y}{0+y^2} = \lim_{y\to 0} 0 = 0$$

(iii) along $y=x$:

$$\lim_{x\to 0} \frac{2x(x)}{3x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{4x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$$0 \neq \frac{1}{2}$$

Since on different paths ~~to~~ passing through $(0,0)$, we get different limits, then

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+y^2}$$ does not exist $\nexists$.

<u>Def^n</u>: $f(x,y)$ is cont. at (x_0, y_0) if

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$. $f(x,y)$ is cont. if it's

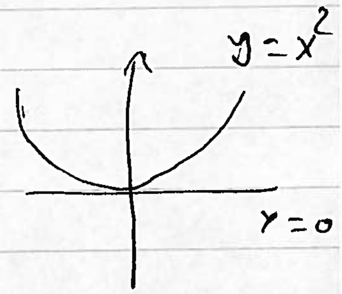
cont. at each point of its domain.

A similar definition can be adopted for a fn. of three variable

<u>Ex.</u> Find the domain of continuity of

$$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

<u>Soln.</u> $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$



along $y=0$: $\lim\limits_{x\to 0} \dfrac{0}{x^4+0} = 0$

along $y = x^2$: $\lim\limits_{x\to 0} \dfrac{2x^4}{x^4+x^4} = \lim\limits_{x\to 0} 1 = 1$

$0 \neq 1$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2} \nexists$

$\therefore$ f is discontinuous at $(0,0)$.

f is continuous otherwise

$\therefore$ Domain of continuity of f is $\mathbb{R}^2 \setminus \{(0,0)\}$.

Exc. Find $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^4+2y^4+z^2}$

if it exists. (Verify !).

Exc. Let $f(x,y) = \begin{cases} \frac{x^2+y^2}{x^2-y^2} & (x,y)\neq(0,0) \\ K & (x,y)=(0,0) \end{cases}$

Find the value of K so that f is continuous (if any)

14.3 Partial derivatives

Def^n: Let $z = f(x,y)$. The partial derivative of f with respect to x is $f_x = \frac{\partial f}{\partial x}$ (the derivative of f with respect to x by regarding y as a constant.

Also $f_y = \frac{\partial f}{\partial y}$ is the derivative of f with respect to y by regarding x as a constant.

$$f_x = z_x = \frac{\partial z}{\partial x} \quad \& \quad f_y = z_y = \frac{\partial z}{\partial y}.$$

Remark: $f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$ &

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Geometric interpretation: $f_x(a,b)$ is the slope of the tangent line to the curve of intersection of the surface $z = f(x,y)$ & the plane $y = b$.

Similarly, $f_y(a,b)$ is the slope of the tangent line to the curve of intersection of the surface $z = f(x,y)$ & the plane $x = a$.

~~**Remark**: If $F(x,y,z) = C$, where C is a constant,~~
~~then $z_x = -\frac{F_x}{F_y}$ & $z_y = -\frac{F_y}{F_z}$~~

Def^n: $w = f(x,y,z)$. Then

f_x is the derivative of f with respect to x by regarding both y & z as constants.

$$\frac{\partial f}{\partial x} = f_x = w_x = \frac{\partial w}{\partial x}.$$

Similarly, for f_y & f_z.

Remark:

$$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

Remark: If $F(x,y,z) = C$, where C is constant, then $z_x = -\frac{F_x}{F_z}$ & $z_y = -\frac{F_y}{F_z}$.

Def^n: $z = f(x,y)$. Then

$$(z_x)_x = z_{\underrightarrow{xx}} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \& \quad z_{\underrightarrow{yy}} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = (z_y)_y$$

$$\cancel{z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}$$

$$z_{\underrightarrow{xy}} = (z_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\underleftarrow{\partial y \partial x}}$$

$$z_{\underrightarrow{yx}} = (z_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\underleftarrow{\partial x \partial y}}$$

$$z_{\underrightarrow{xyx}} = (z_{xy})_x = \frac{\partial^3 z}{\underleftarrow{\partial x \partial y \partial x}} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y \partial x}\right)$$

We define higher partial derivatives of $w = f(x,y,z)$ in a similar way as above.

Ex. $f(x,y) = e^{xy}$. Find

$f_x(2,1)$ & $f_y(2,1)$ & $f_{xx}(2,1)$ & $f_{xy}(2,1)$

$f_x = y e^{xy}$

$f_x(2,1) = 1\, e^{(2)(1)} = e^2.$

$f_y = x e^{xy}$

$f_y(2,1) = 2\, e^{(2)(1)} = 2e^2.$

$f_{xx} = (f_x)_x = y^2 e^{xy} \Longrightarrow f_{xx}(2,1) = e^2.$

$f_{xy} = (f_x)_y =$

$xy\, e^{xy} + e^{xy}$

$\therefore f_{xy}(2,1) = 2e^2 + e^2 = 3e^2.$

Ex. $w = (2x+3y-4z)^5.$

Find $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$

Soln. w_{zyx}

$w_z = -20(2x+3y-4z)^4.$

$w_{zy} = (w_z)_y = -240(2x+3y-4z)^3.$

$w_{zyx} = (w_{zy})_x = (-240)(2)(3)(2x+3y-4z)^2$

$= -1440(2x+3y-4z)^2$

Ex. Consider the ellipsoid $x^2+4z^2+2y^2=5$.

Find the rate of change of z with respect to y at $(1,0,-1)$

Soln. $z_y = ?$

Method I : Implicit differentiation

$$x^2 + 4z^2 + 2y^2 = 5$$ $\underline{u}$

Differentiate with respect to y.

$$0 + 8zz_y + 4y = 0$$

$$8zz_y = -4y \Rightarrow z_y = \frac{-y}{2z}$$

$$z_y\Big|_{(1,0,-1)} = \frac{0}{-2} = 0.$$

<u>Method II</u> : $z_y = -\dfrac{F_y}{F_z}$, where

$$F(x,y,z) = x^2 + 4z^2 + 2y^2.$$

$$F_y = 4y$$

$$F_z = 8z$$

So $z_y = -\dfrac{F_y}{F_z} = -\dfrac{4y}{8z} = \dfrac{-y}{2z}$

<u>Ex.</u> $z = \dfrac{x+y}{x-y}$

Find z_{xx} & z_{xy} & z_{yy}.

$$\cancel{z_x = \frac{x(x-y) - xy(1-0)}{(x-y)^2} = \frac{x^2 - 2xy}{(x-y)^2}}$$

$$z_x = \frac{x-y-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$z_x = \frac{-2y}{(x-y)^2}$$

$$\underset{\rightarrow}{z_{xx}} = (z_x)_x = \frac{+2y\,(2)(x-y)(1-0)}{(x-y)^4} = \frac{4y}{(x-y)^3}.$$

$$\underset{\rightarrow}{z_{xy}} = (z_x)_y = \frac{-2(x-y)^2 + 2y(2)(x-y)(-1)}{(x-y)^4}$$

$$= \frac{-2(x-y)^2 - 4y(x-y)}{(x-y)^4}$$

$$= \frac{-2(x-y) - 4y}{(x-y)^4}$$

$$= \frac{-2x-2y}{(x-y)^3}.$$

$$= -\frac{2(x+y)}{(x-y)^3}$$

$$z = \frac{x+y}{x-y}$$

$$z_y = \frac{(x-y) - (x+y)(0-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

$$\underset{\rightarrow}{z_{yy}} = (z_y)_y = \frac{-2x(2)(x-y)(0-1)}{(x-y)^4} = \frac{4x}{(x-y)^3}$$

Ex. $f(x,y) = \begin{cases} \dfrac{x}{x+y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

Find $f_y(0,0)$ if exists.

Soln. $$f_y(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{h+0}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \qquad \nexists$$

Hence, $f_y(0,0)$ does not exist.

✓ Exc. $f(x,y) = \begin{cases} \dfrac{y^2}{x+y} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Find $f_x(0,0)$ if it exists.

Exc. Let $x^2 - \ln(y-z) = 3y^2$.

Find z_y & z_x.

14.4 Tangent planes

Th. The surface $z = f(x,y)$ has as a normal $n = \langle z_x, z_y, -1\rangle$, that is, n is normal to the tangent plane to the surface.

Th. The surface $F(x,y,z) = C$ has as a normal $n = \langle F_x, F_y, F_z \rangle$.

Ex. Find the equation of the tangent ~~line~~ plane ~~to~~ and the normal line to the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ at $(1,-1,1)$

Soln. $F(x,y,z) = x^2 + 2y^2 + 3z^2$

$F_x = 2x = 2(1) = 2$

$F_y = 4y = 4(-1) = -4$

$F_z = 6z = 6(1) = 6$.

$\therefore\ n = \langle F_x, F_y, F_z\rangle = \langle 2,-4,6\rangle$ is normal to the surface at $(1,-1,1)$.

(You can deal with $\langle 1,-2,3\rangle$ as a normal)

$\therefore$ the equation of the tangent plane to the surface at $(1,-1,1)$ is

$2(x-1) + -4(y+1) + 6(z-1) = 0$.

$x - 1 - 2(y+1) + 3(z-1) = 0$

$\boxed{x - 2y + 3z = 6}$. Equations of the normal line are:

$x = 2t+1$, $y = -4t-1$, $z = 6t+1$

Ex. Find the eqn. of the tangent ~~line~~ plane to the surface $z = x e^{xy}$ at $(1,0)$.

$$z_x = xy e^{xy} + e^{xy} = (1)(0)\, e^{(1)(0)} + e^{(1)(0)} = 1$$

$$z_y = x^2 e^{xy} = 1\, e^{(1)(0)} = 1$$

$\therefore\ n = \langle z_x, z_y, -1 \rangle = \langle 1, 1, -1 \rangle$ is normal to the surface.

at $(1,0)$, $z = 1\, e^{(1)(0)} = 1$

$\therefore$ the eqn. of the tangent ~~line~~ plane at $(1,0,1)$ is

$$1(x-1) + 1(y-0) - 1(z-1) = 0$$

$$\boxed{x + y - z = 0}$$

Differentials: If $z = f(x,y)$, the differential of z is:

$$dz = f_x\, dx + f_y\, dy.$$

If x varies from a to x &
y varies from b to y, then

$$\Delta x = x - a$$
$$\Delta y = y - b$$

$\Delta z = f(x,y) - f(a,b)$ and the linear approximation of f at (a,b) is:

$\Delta z \approx dz$ where $dx = \Delta x \approx 0$ & $dy = \Delta y \approx 0$, i.e.

$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$ where $x \approx a$ & $y \approx b$.

The linear approximation of $f(x,y,z)$ at (a,b,c) is: $f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$; $x \approx a$, $y \approx b$, $z \approx$

Ex. Use the linear approximation to approximate $\underline{3}$

$$\sqrt{(3.02)^2 + (3.97)^2}$$

Soln. $f(x,y) = \sqrt{x^2 + y^2}$.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad , \quad \begin{matrix} x \approx a, \\ y \approx b. \end{matrix}$$

$x = 3.02 \quad , \quad y = 3.97$

$a = 3 \quad , \quad b = 4$

$$f_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_x(3,4) = \frac{3}{\sqrt{9+16}} = \frac{3}{\sqrt{25}} = \frac{3}{5} .$$

$$f_y = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_y(3,4) = \frac{4}{\sqrt{9+16}} = \frac{4}{5} .$$

$f(3,4) = \sqrt{9+16} = 5$

$$f(x,y) \approx f(3,4) + f_x(3,4)(0.02) + f_y(3,4)(-0.03)$$

$$\approx 5 + \frac{3}{5}(0.02) + \frac{4}{5}(-0.03)$$

$$= 5 + \frac{0.06}{5} - \frac{\overset{0.12}{\cancel{0.06}}}{5}$$

$$= 5 - \frac{0.06}{5}$$

$$\therefore \sqrt{(3.02)^2 + (3.97)^2} \approx 5 - \frac{0.06}{5}$$

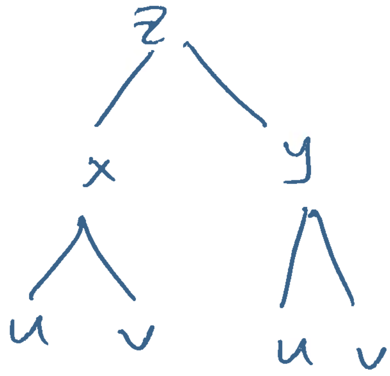
14.5 Chain Rule

$\boxed{1}$

$z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$ Then

$z_u = z_x x_u + z_y y_u$ &

$z_v = z_x x_v + z_y y_v$.



Ex. let $z = f(x,y)$,

$x = u - v$, $y = v - u$.

Show that $\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} = 0$.

Show $z_u = z_x x_u + z_y y_u$

$= z_x (1) + z_y (-1)$

$= z_x - z_y$.

$z_v = z_x x_v + z_y y_v$

$= z_x (-1) + z_y (1)$

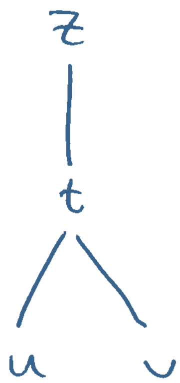
$= -z_x + z_y$.

$\therefore z_u + z_v = \cancel{z_x} - \cancel{z_y} - \cancel{z_x} + \cancel{z_y} = 0$.

$\boxed{2}$ $z = f(t)$, $t = t(u, v)$

$$z_u = \frac{dz}{dt} t_u \quad \&$$

$$z_v = \frac{dz}{dt} t_v .$$



Ex. $z = f(x, y)$,

$x = u - v$, $y = v - u$.

Show $\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} = 0$.

Solu. let $t = u - v$. Then

$$z = f(x, y) = f(t, -t) = f(t).$$

$$z_u = \frac{dz}{dt} t_u = \frac{dz}{dt}(1) = \frac{dz}{dt}.$$

$$z_v = \frac{dz}{dt} t_v = \frac{dz}{dt}(-1) = -\frac{dz}{dt}.$$

$$\therefore z_u + z_v = \frac{dz}{dt} - \frac{dz}{dt} = 0.$$

Ex. $z = f(x, y)$

$$x = u^2 - v^2, \quad y = v^2 - u^2 .$$

Show $\cancel{x \dfrac{\partial z}{\partial y} + y \dfrac{\partial z}{\partial x} = 0}$. $u \dfrac{\partial z}{\partial v} + v \dfrac{\partial z}{\partial u} = 0 .$

Soln.

$$z_v = z_x \, x_v + z_y \, y_v$$

$$= z_x(-2v) + z_y(2v)$$

$$= 2v\, z_y - 2v\, z_x .$$

$$z_u = z_x \, x_u + z_y \, y_u$$

$$= z_x(2u) + z_y(-2u) .$$

$$= 2u\, z_x - 2u\, z_y .$$

$$u\, z_v + v\, z_u = u(2v\, z_y - 2v\, z_x) + v(2u\, z_x - 2u\, z_y)$$

$$= \cancel{2uv\, z_y} - \cancel{2uv\, z_x} + \cancel{2uv\, z_x} - \cancel{2uv\, z_y}$$

$$= 0 .$$

Another Method: let $t = u^2 - v^2$. Then

$$z = f(t, -t) = f(t) .$$

$$z_v = \frac{dz}{dt} t_v = \frac{dz}{dt}(-2v)$$

$$z_u = \frac{dz}{dt} t_u = \frac{dz}{dt}(2u)$$

$$\therefore \quad u z_v + v z_u$$

$$= u\left(-2v \frac{dz}{dt}\right) + v\left(2u \frac{dz}{dt}\right)$$

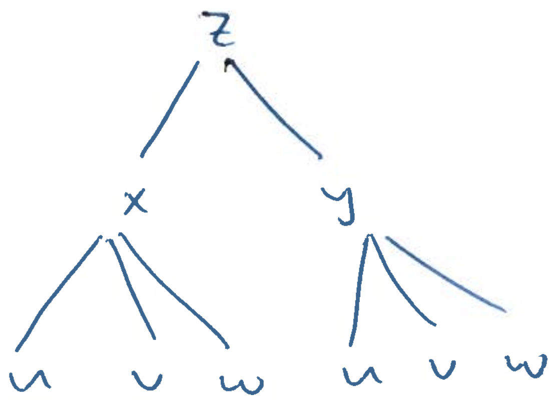
$$= -2uv \frac{dz}{dt} + 2uv \frac{dz}{dt} = 0.$$

Remark: The above formulas of the chain rule may be extended to more variables as seen from the following trees:

$\boxed{3}$ $z_u = z_x x_u + z_y y_u$

$z_v = z_x x_v + z_y y_v$

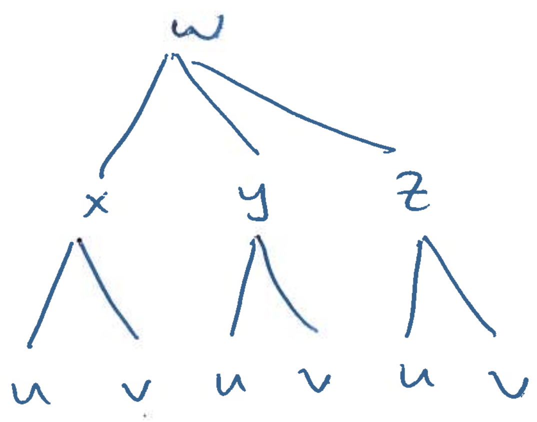
$z_w = z_x x_w + z_y y_w$.



$\boxed{4}$

$w_u = w_x x_u + w_y y_u + w_z z_u$ &

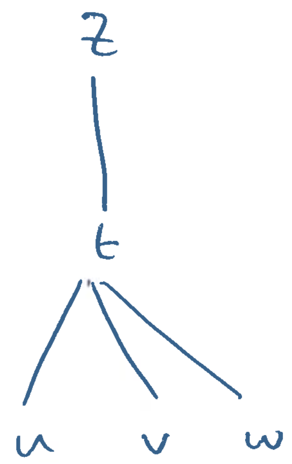
$w_v = w_x x_v + w_y y_v + w_z z_v$



$\boxed{5}$ $z_u = \frac{dz}{dt} t_u$

$z_v = \frac{dz}{dt} t_v$

$z_w = \frac{dz}{dt} t_w$.

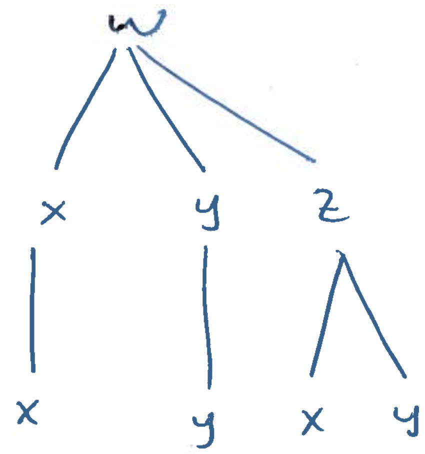


* Remember that if $F(x,y,z) = C$. Then

$z_x = -\frac{F_x}{F_z}$ & $z_y = -\frac{F_y}{F_z}$

We will prove this fact now.

<u>Proof</u>: let $w = F(x,y,z)$



$\cancel{w_x = w_x \frac{dx}{dx} + w_y}$

$w_x = w_x x_x + w_y y_x + w_z z_x$

$0 = w_x (1) + \cancel{w_y (0)} + w_z z_x$ (y is independent from x).

$0 = w_x + w_z z_x$

$\cancel{w}$ $\boxed{z_x = -\frac{w_x}{w_z}}$.

Simillarly, you get that $z_y = -\frac{w_y}{w_z}$.

* let $F(x,y) = C$. Then

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

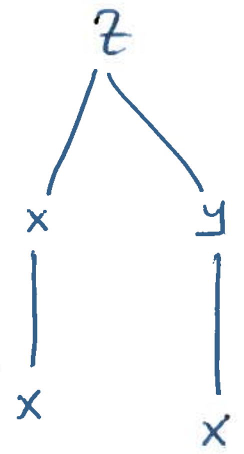
. This is a direct method to find $\frac{dy}{dx}$ instead of using implicit differentiation to find the derivative.

Proof : let $z = f(x,y)$.

$$z_x = z_x\, x_x + z_y\, y_x$$

$$0 = z_x(1) + z_y \frac{dy}{dx} \quad \text{(y depends on x)}$$

$$\frac{dy}{dx} = -\frac{z_x}{z_y}$$



Ex. Find y' if $x^3 + y^3 = 6xy$.

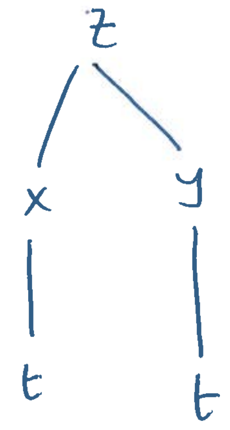
$$x^3 + y^3 - 6xy = 0$$

$$F(x,y) = x^3 + y^3 - 6xy.$$

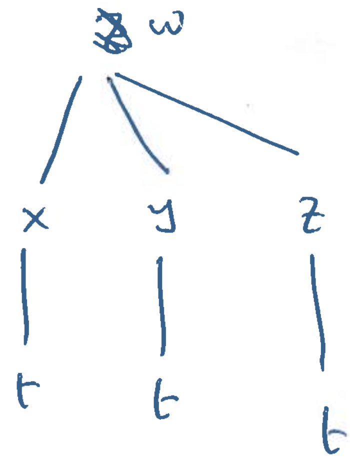
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

$$= -\frac{x^2 - 2y}{y^2 - 2x}$$

$\boxed{6}$ $$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$



$\boxed{7}$ $$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$$



Ex. $w = x^2 y + 3xy^4 + \sqrt{z+3}$

$x = \sin 2t, \; y = \cos t, \; z = e^{2t}$.

Find $\frac{dw}{dt}$ when $t = 0$.

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$$

$$= \cancel{2xy(2\cos 2t) +}$$

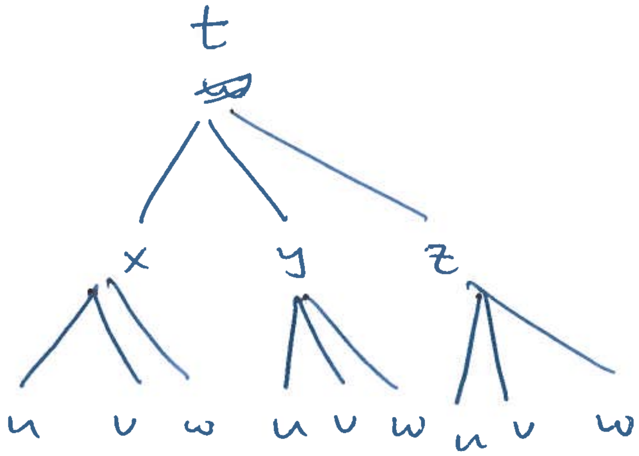
$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t) + \frac{1}{2\sqrt{z+3}}(2e^{2t}).$$

$t = 0 \longrightarrow x = 0, y = 1, z = 1$

$$\left.\frac{dw}{dt}\right|_{t=0} = (0+3)(2) + (0+0)(0) + \frac{1}{4}(2)$$

$$= \frac{13}{2}.$$

$\boxed{8}$



$\cancel{w_u = w_x x_u + w_y y_u + w_z z_u}$

$\cancel{w_v = w_x x_v + w_y y_v + w_z z_v}$

$$t_u = t_x x_u + t_y y_u + t_z z_u$$

$$t_v = t_x x_v + t_y y_v + t_z z_v$$

$$t_w = t_x x_w + t_y y_w + t_z z_w.$$

Ex. $w = xy + yz + xz$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$

Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial \theta}$ when $r = 2$, $\theta = \pi/2$.

$$w_r = w_x x_r + w_y y_r + w_z z_r$$

$$= (y+z)\cos\theta + (x+z)\sin\theta + (y+x)\theta$$

$r = 2$, $\theta = \pi/2 \Rightarrow x = 0$, $y = 2$, $z = \pi$.

$$w_r = (2+\cancel{\pi})(0) + (0+\pi)(1) + (0+2)(\pi/2) = \pi + \pi = 2\pi.$$

$$w_\theta = w_x x_\theta + w_y y_\theta + w_z z_\theta$$

$$= (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r)$$

$$= (2+\pi)(-2) + (0+\cancel{\pi})(0) + (0+2)(2)$$

$$= -2(2+\pi) + 4$$

$$= -4 - 2\pi + 4 = -2\pi.$$

Exc. let $z = f(x^2+y^2, x^2-y^2)$.

Find z_{xx} & z_{yy}.

14.6 Directional derivative and the gradient vector.

Def^n: The directional derivative of $f(x,y)$ in the direction of a unit vector $u = \langle a, b \rangle$ is:

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b.$$

Def^n: The directional derivative of $f(x,y,z)$ in the direction of a unit vector $u = \langle a, b, c \rangle$ is:

$$D_u f(x,y,z) = f_x(x,y,z)\,a + f_y(x,y,z)\,b + f_z(x,y,z)\,c.$$

Def^n: The gradient of $f(x,y)$ is:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle.$$

Def^n: The gradient of $f(x,y,z)$ is:

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle.$$

Remark

1) $D_u f(x,y) = \nabla f(x,y) \cdot u$.

2) $D_u f(x,y,z) = \nabla f(x,y,z) \cdot u$.

Remark

1) The maximum value of $D_u f(x,y)$ is $|\nabla f(x,y)|$ and it occurs when u has the same direction as $\nabla f(x,y)$.

2) The minimum value of $D_u f(x,y)$ is $-|\nabla f(x,y)|$ and it occurs when u has the opposite direction as $+\nabla f(x,y)$, i.e. when u has the direction of $-\nabla f(x,y)$.

Remark

1) The maximum value of $D_u f(x,y,z)$ is $|\nabla f(x,y,z)|$ and it occurs when u has the same direction as $\nabla f(x,y,z)$, ~~i.e. when u has the direction of~~ $\cancel{-\nabla f(x}$

2) The minimum value of $D_u f(x,y,z)$ is $-|\nabla f(x,y,z)|$ and it occurs when u has the opposite direction as $\nabla f(x,y,z)$, i.e. when u has the direction of $-\nabla f(x,y,z)$.

Ex. Find ~~the directional derivative~~, gradient, the maximum rate of change, the minimum rate of change of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $(3,6,-2)$. Find also the direction in which f increases most rapidly at $(3,6,-2)$ and the direction in which f decreases most rapidly at $(3,6,-2)$. Find also the directional derivative of f at $(3,6,-2)$ in the direction of $(1,5,-4$

Soln.
$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{3}{7}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{6}{7}.$$

$$f_z = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{-2}{7}.$$

1) $\nabla f(3,6,-2) = \langle f_x, f_y, f_z \rangle = \langle 3/7, \frac{6}{7}, -2/7 \rangle$

2) max. rate of change $= |\nabla f| = \cancel{8}\ \frac{1}{7}\sqrt{9+36+4} = 1$

3) min. rate of change $= -|\nabla f| = -1$

4) The direction in which f increases most rapidly is

$$\nabla f = \frac{1}{7}\langle 3, 6, -2 \rangle.$$

5) The direction in which f decreases most rapidly is

$$-\nabla f = -\frac{1}{7}\langle 3, 6, -2 \rangle.$$

6) $a = \langle -2, -1, -2 \rangle$

$|a| = \sqrt{4+1+4} = 3.$

$u = \frac{a}{|a|} = \frac{1}{3}\langle -2, -1, -2 \rangle$

$= \langle -\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \rangle. = \langle u_1, u_2, u_3 \rangle$

$D_u f = f_x\, u_1 + f_y\, u_2 + f_z\, u_3$

$= \frac{3}{7}\left(-\frac{2}{3}\right) + \frac{6}{7}\left(-\frac{1}{3}\right) + \frac{-2}{7}\left(-\frac{2}{3}\right)$

$= \frac{-6}{21} - \frac{6}{21} + \frac{4}{21} = \frac{-8}{21}$

Remark: 1) max. rate of change of f = the largest directional derivative of f

2) min. rate of change of f = the smallest directional derivative of f.

3) rate of change of f in the direction of u = $D_u f$.

4) f increase most rapidly = $D_u f$ is the largest.

5) f decreases most rapidly = $D_u f$ is the smallest.

Exc. $f(x,y) = \sin(xy)$, $(1,0)$.

Find the maximum rate of change of f at the given point and the direction in which it occurs.

Find also the minimum rate of change of f at the given point and the direction in which it occurs.

14.7 Maximum & Minimum Values

Def$^{\underline{n}}$: 1) $f(x,y)$ has a local max. at (a,b) if $f(x,y) \le f(a,b)$ $\forall (x,y)$ in an open disc centered at (a,b).
$f(a,b)$ is called local max. value.

2) $f(x,y)$ has a local min. at (a,b) if
$f(x,y) \ge f(a,b)$ $\forall (x,y)$ in an open disc centered at $(a,b$
$f(a,b)$ is called local min. value.

3) $f(x,y)$ has an absolute max. at (a,b) if
$f(x,y) \le f(a,b)$ $\forall (x,y)$ in the domain of f.
$f(a,b)$ is called absolute max. value.

4) $f(x,y)$ has an absolute min. at (a,b) if
$f(x,y) \ge f(a,b)$ $\forall (x,y)$ in the domain of f
$f(a,b)$ is called absolute min. value.

5) (a,b) is called a critical point of $f(x,y)$ if
$f_x(a,b) \nexists$ $\boxed{\text{or}}$ $f_y(a,b) \nexists$ or
$f_x(a,b) = f_y(a,b) = 0$.

Remark: Functions we will deal with have first order partial derivatives. Therefore, (a,b) critical point of $f(x,y)$ will mean that

$$\boxed{f_x(a,b) = f_y(a,b) = 0}.$$

Th. If f has a local max. or min. at (a,b), then (a,b) is a critical point of $f(x,y)$.

Remark: The converse of the above theorem need not be true.

Th. (Second derivative test).

If $f(x,y)$ has continuous second partial derivatives on a disk with center (a,b) &

$f_x(a,b) = f_y(a,b) = 0$. (i.e. (a,b) is a critical point of f).

let $D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b)$

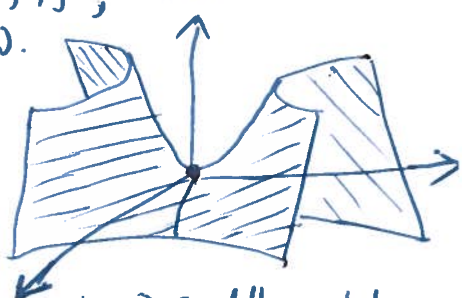
1) If $D > 0$, $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max

2) If $D > 0$, $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.

3) If $D < 0$, then $f(a,b)$ is not a local max. or min. &

(a,b) is called a سرجية saddle point of $f(x,y)$, i.e. the graph of $z = f(x,y)$ crosses its tangent plane at (a,b).

4) If $D = 0$, the test fails.



Ex. Find the local max. and min. values and saddle points of f, if any.

1) $f(x,y) = 2x - 2xy - 2y$.

$f_x = 2 - 2y = 0 \longrightarrow y = 1$

$f_y = -2x - 2 = 0 \longrightarrow x = -1$

$\therefore (-1,1)$ is the only critical point of f.

$f_{xx} = 0 \Rightarrow f_{xx}(-1,1) = 0$

$f_{yy} = 0 \Rightarrow f_{yy}(-1,1) = 0$

$f_{xy} = -2 \Rightarrow f_{xy}(-1,1) = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 0 - 4 = -4 < 0$.

$\therefore f$ has a saddle point at $(-1,1)$.

$(-1,1)$ is a saddle point of f.

2) $f(x,y) = x^2 + 2xy + 2y^2$

$f_x = 2x + 2y = 0$

$f_y = 2x + 4y = 0$

$-2y = 0 \longrightarrow y = 0 \longrightarrow x = 0$

$\therefore$ $(0,0)$ is the only critical point of f.

$f_{xx} = 2 \Rightarrow f_{xx}(0,0) = 2$

$f_{yy} = 4 \Rightarrow f_{yy}(0,0) = 4$

$f_{xy} = 2 \Rightarrow f_{xy}(0,0) = 2$.

$D = f_{xx} f_{yy} - f_{xy}^2 = 8 - (2)^2 = 4$

but $f_{xx} = 2 > 0$

So f has a local min. at $(0,0)$.

$f(0,0) = 0 + 0 + 0 = 0$ is a local min. value.

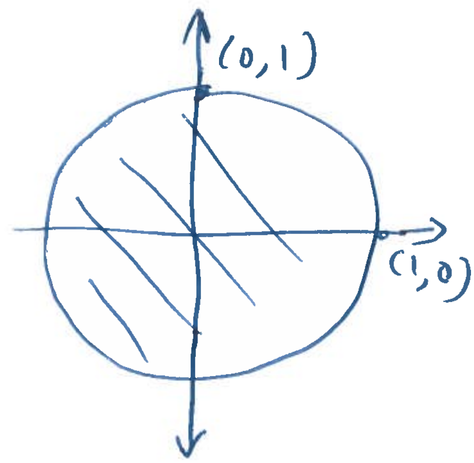
<u>Th.</u> (Extreme Value Th.)

If f continuous on a closed, bounded region D in $\mathbb{R}^2$, then f attains an absolute max. value $f(x_1, y_1)$ and an absolute min. value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

<u>How to find the absolute min. & the absolute max of a cont. fn. $f(x,y)$ over a closed, bounded region D?</u>

1) Find the values of f at the critical points of f <u>in</u> D
2) Find the extreme values of f on the boundary of D
3) The largest of the values from step 1 and 2 is the absolute max. value; the smallest of these values is the absolute min. value.

Ex. Find the absolute max. and min. values of $f(x,y) = x^2 - y - y^2$ on D, where D is the closed disc $x^2 + y^2 \le 1$



Soln. $\boxed{1}$ $f_x = 2x = 0 \longrightarrow x = 0$

$f_y = -1 - 2y = 0 \longrightarrow y = -\frac{1}{2}$.

$\therefore$ $(0, -\frac{1}{2})$ only critical point of f in D

$\boxed{2}$ on D ; $x^2 + y^2 = 1 \longrightarrow$

boundary of $\quad x^2 = 1 - y^2$.

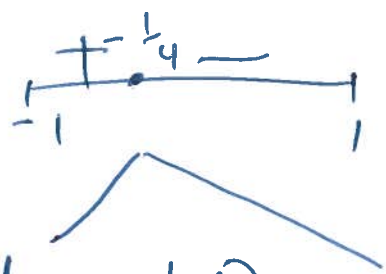
$f(x,y) = x^2 - y - y^2$.

$g(y) = 1 - y^2 - y - y^2$

$g(y) = 1 - y - 2y^2 \qquad y \in [-1, 1]$.

$g'(y) = -1 - 4y \Rightarrow$

$g'(y) = 0 \longrightarrow y = -\frac{1}{4}$



$\therefore$ extreme values of f on the boundary of D occurs at $y = -1, -\frac{1}{4}, 1$

$x^2+y^2=1$

$y=-1 \longrightarrow x=0$

$y=1 \longrightarrow x=0$

$y=-\frac{1}{4} \longrightarrow x^2=\frac{15}{16} \longrightarrow x=\pm\frac{\sqrt{15}}{4}$.

$\boxed{3}$

(x,y)	$(0,-\frac{1}{2})$	~~$(1,-1)$~~	~~$(-1,-1)$~~	$(0,1)$	$(0,-1)$	$(\frac{\sqrt{15}}{4},-\frac{1}{4})$	$(-\frac{\sqrt{15}}{4},-\frac{1}{4})$
$f(x,y)$	$1/4$	~~1~~	~~1~~	-2	0	$9/8$	$9/8$

$f(x,y)=x^2-y-y^2$

$f(0,-\frac{1}{2})=0+\frac{1}{2}-\frac{1}{4}$

$=+1/4$

$f(0,1)=-2$ & $f(0,-1)=0$

$\cancel{f(1,-1)=1+1-1=1}$

$\cancel{f(-1,-1)=1+1-1=1}$

$\cancel{f(1,1)=1-1-1=-1}$

$\cancel{f(-1,1)=1-1-1=-1}$

$f(\frac{\sqrt{15}}{4},-\frac{1}{4})=\frac{15}{16}+\frac{1}{4}-\frac{1}{16}$

$=\frac{15+4-1}{16}=\frac{9}{8}$

$f(\frac{-\sqrt{15}}{4},-\frac{1}{4})=\frac{15}{16}+\frac{1}{4}-\frac{1}{16}=\frac{9}{8}$

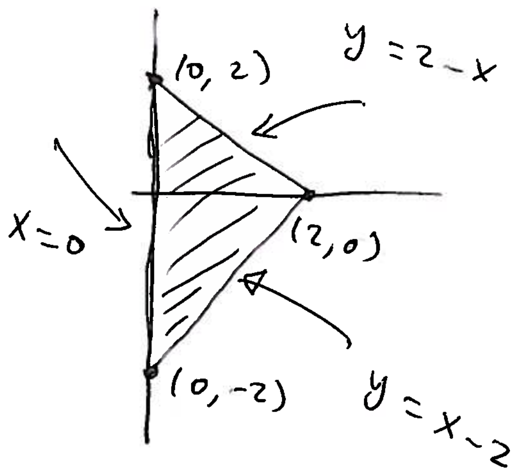
$\therefore$ the absolute max. value of f on D is $9/8$

& " " min. value of f on D is -2

Exc. Find the absolute max. value & the absolute min. value of

$f(x,y) = x + y - xy$ on the closed triangular region D with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.

① find critical points of f inside D.



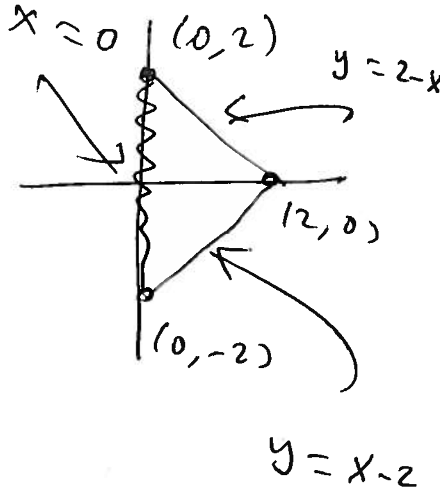
$f_x = 1 - y = 0 \Rightarrow y = 1$

$f_y = 1 - x = 0 \Rightarrow x = 1$

$\therefore$ $\boxed{(1,1)}$ is the critical pt.

Observe $(1,1)$ is not inside D

② We find points of possible local extrema of f on the boundary of D



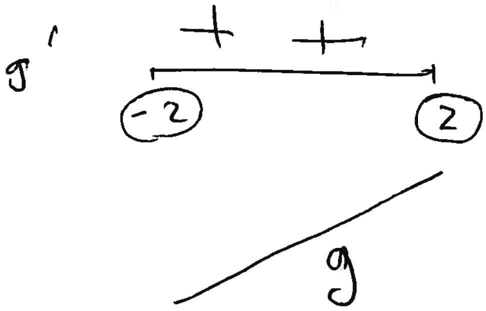
(i) $\boxed{\text{on } \quad x = 0}$

$f(x,y) = x + y - xy$

$= 0 + y - 0y$

$g(y) = y \quad , y \in [-2, 2]$

$g'(y) = 1$



$y = -2 \;, x = 0 \implies \boxed{(0,-2)}$

$y = 2, \; x = 0 \implies \boxed{(0,2).}$

(ii) $y = 2 - x$

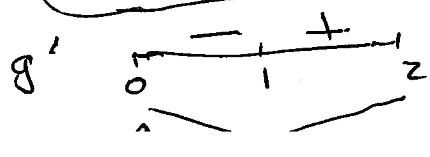
$f(x,y) = x + y - xy$

$= x + (2-x) - x(2-x)$

$= \cancel{x} + 2 - \cancel{x} - 2x + x^2$

$g(x) = 2 - 2x + x^2 \quad , x \in [0,2]$

$g'(x) = -2 + 2x = 0 \implies x = 1$



$y = 2 - x$

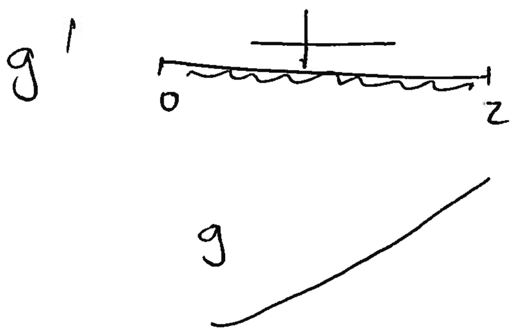
$x = 0 \Longrightarrow y = 2 \Longrightarrow \boxed{(0,2)}$

$x = 1 \Longrightarrow y = 1 \Longrightarrow \boxed{(1,1)}$

$x = 2 \Longrightarrow y = 0 \Longrightarrow \boxed{(2,0)}$

(iii) $y = x - 2$

$$f(x,y) = x + y - xy$$
$$= x + x - 2 - x(x-2)$$
$$= 2x - 2 - x^2 + 2x$$
$$g(x) = 4x - 2 - x^2 \quad , \; x \in [0,2]$$
$$g'(x) = 4 - 2x = 0 \longrightarrow x = 2$$



$y = x - 2$

$x = 0 \Longrightarrow y = -2 \Longrightarrow \boxed{(0,-2)}$

$x = 2 \Longrightarrow y = 0 \Longrightarrow \boxed{(2,0)}$

(x,y)	$(0,-2)$	$(0,2)$	$(1,1)$	$(2,0)$	
$f(x,y)$	-2	2	1	2	

$f(x,y) = x + y - xy$

$f(0,-2) = 0 - 2 - 0 = -2$

$f(0,2) = 0 + 2 - 0 = 2$

$f(1,1) = 1 + 1 - 1 = 1$

$f(2,0) = 2 + 0 - 0 = 2$

The absolute max. value of f is 2, it occurs at $(0,2)$ & $(2,0)$.

The absolute min. value of f is -2, it occurs at $(0,-2)$.

14.8 Lagrange Multipliers

Method of Lagrange Multipliers:

To find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$

1) Find all values of x, y, z and λ such that:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \quad \text{and}$$

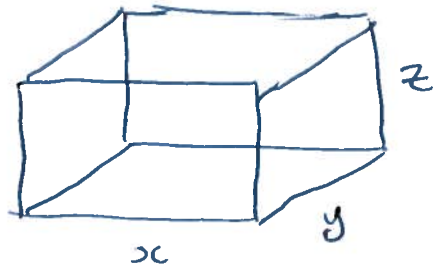
$$g(x,y,z)=k$$

2) Evaluate f at all the points (x,y,z) that result from (1). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

λ is called the Lagrange multiplier.

Ex. A rectangular box without a lid is to be made from $12\ m^2$ cardboard. Find the maximum volume of such a box.

x length
y width
z height



The problem turns out to:

Maximize the volume $V = xyz$ subject to:

$$g(x,y,z) = xy + \cancel{2x+2y}\, 2xz + 2yz = 12$$

We look for values of x, y, z, and λ such that

$$\nabla V = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 12.$$

$$\langle yz, xz, xy \rangle = \lambda \langle y+2z, x+2z, 2x+2y \rangle$$

and $\quad xy + 2xz + 2yz = 12$

$$\therefore \quad yz = \lambda(y+2z) \quad \text{—— (1)}$$
$$xz = \lambda(x+2z) \quad \text{—— (2)}$$
$$xy = \lambda(2x+2y) \quad \text{—— (3)}$$
$$xy + 2xz + 2yz = 12 \quad \text{—— (4)}$$

$$\therefore \quad xyz = \lambda(xy + 2xz) \quad \text{—— (5)}$$
$$xyz = \lambda(xy + 2yz) \quad \text{—— (6)}$$
$$xyz = \lambda(2xz + 2yz) \quad \text{—— (7)}$$

Now $\lambda \neq 0$ because if $\lambda = 0$, then from (1), (2) and (3), $yz = xz = xy = 0$ which contradicts (4).

$\therefore \lambda \neq 0$

From (5) and (6),

$$\lambda(xy + 2xz) = \lambda(xy + 2yz)$$

$$\therefore xy + 2xz = xy + 2yz$$

$$\therefore 2xz = 2yz$$

$$\therefore xz = yz$$

but $z \neq 0$ because if $z = 0$, then $V = 0$

$$\therefore z \neq 0$$

$$\therefore x = y$$

From (6) and (7),

$$\lambda(xy + 2yz) = \lambda(2xz + 2yz)$$

$$\therefore xy + \cancel{2yz} = 2xz + \cancel{2yz}$$

$$\therefore xy = 2xz$$

but $x \neq 0$ because if $x = 0$, then $V = 0$

$$\therefore y = 2z$$

$$\therefore x = y = 2z$$

From (4) ; $xy + 2xz + 2yz = 12$, we get

$$(2z)(2z) + (2)(2z)(z) + 2(2z)(z) = 12$$

$$\therefore 4z^2 + 4z^2 + 4z^2 = 12$$

$$\therefore 12z^2 = 12 \Rightarrow z^2 = 1 \Rightarrow z = 1$$

~~Thus $V = xyz =$~~

$\therefore \quad x = 2z = 2(1) = 2$

$y = 2z = 2.$

$\therefore \quad V = xyz$

$= (2)(2)(1)$

$= 4$

$\therefore$ the maximum volume of the box is $4 \ m^3$.