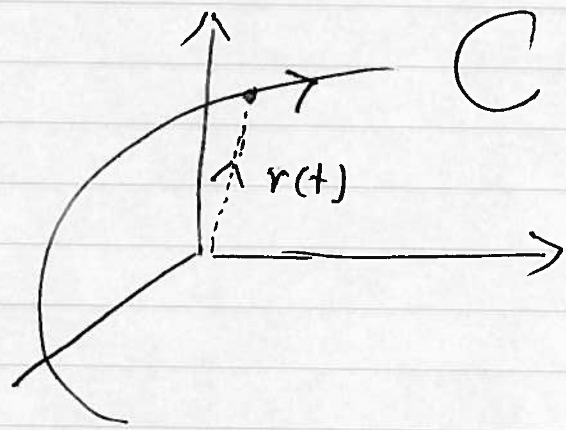
13.1 Vector ~~valued~~ fn's & Space curves

$r(t) = \langle x(t), y(t), z(t) \rangle$ is called a vector valued fn.



C: curve of r(t)

$$\lim_{t \to a} r(t) = \langle \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \rangle$$

Ex: $\displaystyle \lim_{t \to 1} \left(\frac{t^2-1}{t-1} i + \sqrt{t+8}\, j + \frac{\sin \pi t}{\ln t} k \right)$

$$= \langle \lim_{t \to 1} \frac{t^2-1}{t-1}, \lim_{t \to 1} \sqrt{t+8}, \lim_{t \to 1} \frac{\sin \pi t}{\ln t} \rangle$$

$$= \langle \lim_{t \to 1} t+1, \sqrt{1+8}, \lim_{t \to 1} \frac{\pi \cos \pi t}{\frac{1}{t}} \rangle$$

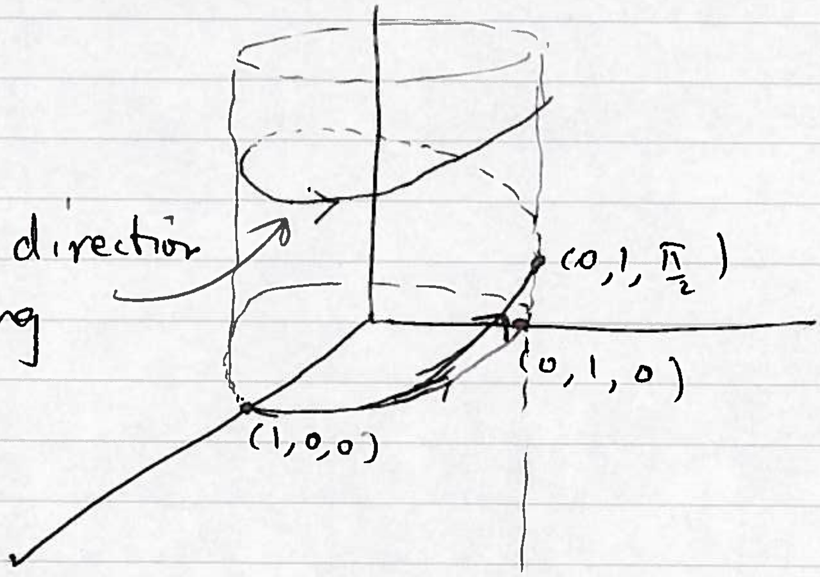
$$= \langle 2, 3, \lim_{t \to 1} \pi t \cos \pi t \rangle$$

$$= \langle 2, 3, -\pi \rangle$$

Ex. Sketch $r(t) = \langle \cos t, \sin t, t \rangle$

The curve is called a helix (circular helix)

$x^2 + y^2 = 1$



* $r(t) = \langle a\cos t, b\sin t, ct \rangle$

(curve : elliptic helix)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

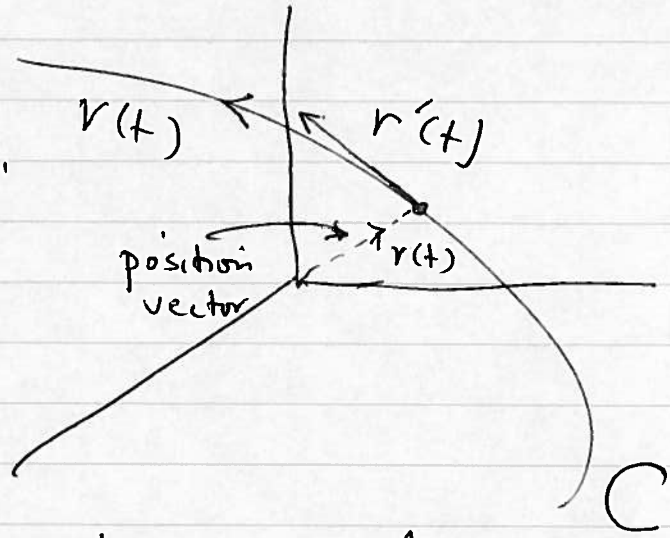
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (elliptic cyllinder)}$$

Exc. Sketch the curve of the vector function $r(t) = \langle \cos t, \sin t, 1 \rangle$. Also identify the curve and ~~identify~~ indicate the direction of increasing parameter.

Derivative & integrals of vector functions

13.2 $r(t) = \langle x(t), y(t), z(t) \rangle$

$$r'(t) = \langle x'(t), y'(t), z'(t) \rangle$$



unit tangent vector to C at t:

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Ex. $r(t) = \langle t\sin t, t^2, \frac{1}{t^2+1} \rangle$. Find the unit tangent vector to C at $t = 0$

$$r'(t) = \langle t\cos t + \sin t, 2t, \frac{-1}{(t^2+1)^2} \rangle$$

$$r'(0) = \langle 0\cos 0 + \sin 0, 2(0), -1 \rangle$$
$$= \langle 0, 0, -1 \rangle \implies |r'(0)| = \sqrt{0+0+1} = 1$$

$$T(t) = \frac{r'(0)}{|r'(0)|} = \frac{\langle 0, 0, -1 \rangle}{1}$$

$$= \langle 0, 0, -1 \rangle = -k.$$

$\ast \quad r(t) = \langle x(t), y(t), z(t) \rangle$

Consider the constant of integration to be zero.

$$\int r(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle$$

$$\int_a^b r(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

<u>Ex</u> $\displaystyle \int_0^1 \left(\frac{4}{1+t^2}\, j + \frac{2t}{1+t^2}\, k \right) dt$

$$= \int_0^1 \left\langle 0, \frac{4}{1+t^2}, \frac{2t}{1+t^2} \right\rangle dt$$

$$= \left\langle \int_0^1 0\,dt, \int_0^1 \frac{4}{1+t^2}\,dt, \int_0^1 \frac{2t}{1+t^2}\,dt \right\rangle$$

$$= \left\langle 0(1-0), \; 4\tan^{-1} t \Big]_0^1, \; \ln(1+t^2) \Big]_0^1 \right\rangle$$

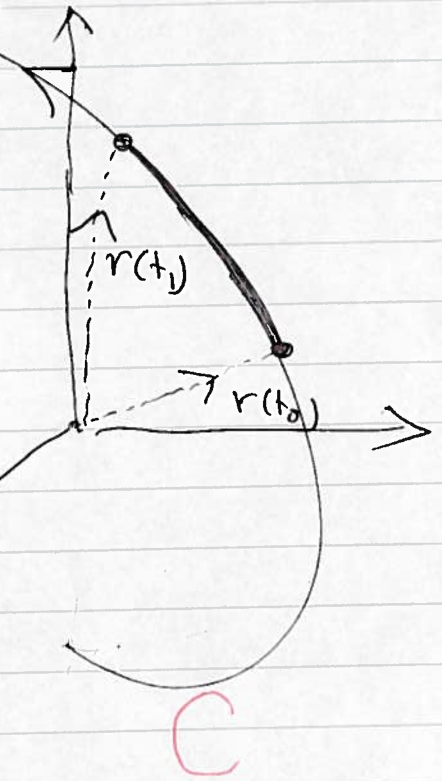
$$= \left\langle 0, \; 4(\tan^{-1} 1 - \tan^{-1} 0), \; \ln(1+1) - \ln(1+0) \right\rangle$$

$$= \left\langle 0, \; 4\left(\frac{\pi}{4} - 0\right), \; \ln 2 - \cancel{\ln 1} \right\rangle$$

$$= \langle 0, \pi, \ln 2 \rangle$$

$$= \pi j + \ln 2\, k.$$

13.3 Arc length & Curvature



Th. $r(t) = \langle x(t), y(t), z(t) \rangle$

The length of the arc from t_0 to t_1 on C is:

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

Ex. Find the length of the curve

$r(t) = \langle \sin t, \cos t, \cancel{\sec} t \rangle, \quad 0 \le t \le \pi/4.$

$r'(t) = \langle \cos t, -\sin t, \cancel{\sec t}\ 1 \rangle$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{\cos^2 t + \sin^2 t + \cancel{\sec^4 t}\ 1}\, dt$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \cancel{\sec^4 t}\ 1}\, dt = \int_0^{\frac{\pi}{4}} \sqrt{2}\, dt$$

$$= \sqrt{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\sqrt{2}\,\pi}{4}.$$

<u>Def^n</u>: Let $r(t)$ be a vector fn. The curvature $\kappa(t)$ of C is:

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|}$$, where $T(t)$ is the unit tangent vector.

$$= \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

<u>Ex</u>. Let $r(t) = \langle \cos t, \sin t, t \rangle$.

Find the curvature of C at the point $(0, 1, \frac{\pi}{2})$

<u>Soln</u> $t = \pi/2$.

$r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.

$T(t) = \dfrac{r'(t)}{|r'(t)|} = \dfrac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$.

$T'(t) = \dfrac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$

$T'(\pi/2) = \dfrac{1}{\sqrt{2}} \langle 0, -1, 0 \rangle$

$|T'(\frac{\pi}{2})| = \dfrac{1}{\sqrt{2}} (1) = \dfrac{1}{\sqrt{2}}$.

$$r'(\pi/_2) = \langle -1, 0, 1 \rangle$$

$$|r'(\pi/_2)| = \sqrt{2}\,.$$

$$\therefore \kappa(\pi/_2) = \frac{|T'(\pi/_2)|}{|r'(\pi/_2)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}\,.$$

or to say :

$$r'(\pi/_2) = \langle -1, 0, 1 \rangle.$$

$$r''(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$r''(\tfrac{\pi}{2}) = \langle 0, -1, 0 \rangle.$$

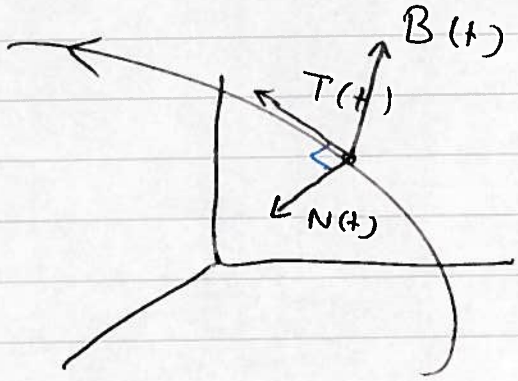
$$r'(\tfrac{\pi}{2}) \times r''(\tfrac{\pi}{2}) = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \langle 1, 0, 1 \rangle$$

$$|r'(\tfrac{\pi}{2}) \times r''(\tfrac{\pi}{2})| = \sqrt{2}\,.$$

$$\therefore \kappa(\tfrac{\pi}{2}) = \frac{|r'(\frac{\pi}{2}) \times r''(\frac{\pi}{2})|}{|r'(\frac{\pi}{2})|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

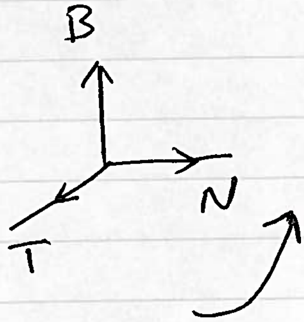
Def^n: Let $r(t)$ be a vector fn. The unit normal vector to C is



$$N(t) = \frac{T'(t)}{|T'(t)|} \quad \&$$

the binormal vector to C is

$$B(t) = T(t) \times N(t)$$



Remark: $B(t)$ is a unit vector

Remark: T, N and B behave like i, j and k, that is:

$$T \times N = B, \quad N \times B = T \ \& \ B \times T = N.$$

Ex. $r(t) = \langle \cos t, \sin t, t \rangle$.

Find the unit normal vector and the binormal vector to C at $(0, 1, \frac{\pi}{2})$.

Soln. $t = \pi/2$.

We have seen in the previous example that

$$T'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

$$T'(\tfrac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle 0, -1, 0 \rangle$$

$$|T'(\tfrac{\pi}{2})| = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

$$\therefore N(\tfrac{\pi}{2}) = \frac{T'(\frac{\pi}{2})}{|T'(\frac{\pi}{2})|} = \frac{\cancel{\frac{1}{\sqrt{2}}}\langle 0, -1, 0 \rangle}{\cancel{\frac{1}{\sqrt{2}}}}$$

$$= \langle 0, -1, 0 \rangle = -j$$

Also from the previous example

$$T(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$$

$$T(\tfrac{\pi}{2}) = \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle$$

$$\therefore B(\tfrac{\pi}{2}) = T(\tfrac{\pi}{2}) \times N(\tfrac{\pi}{2})$$

$$= \frac{1}{\sqrt{2}} \langle -1, 0, 1 \rangle \times \langle 0, -1, 0 \rangle$$

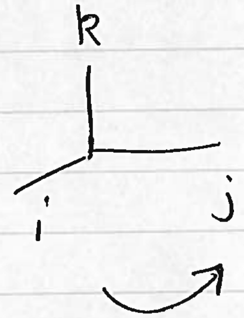
$$= \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle$$

You can say immediately:

$$B\left(\tfrac{\pi}{2}\right) = T\left(\tfrac{\pi}{2}\right) \times N\left(\tfrac{\pi}{2}\right)$$

$$= \frac{1}{\sqrt{2}}(-i+k) \times -j$$



$$= \frac{1}{\sqrt{2}}(k+i)$$

$$= \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle.$$

Exc 1 let $r(t) = \langle 2\cos t, 2\sin t, 1\rangle$

Find $\kappa(t)$ of C. Describe the curve C.

Exc 2 let $r(t) = \langle 2+t, 3-t, 2t+1\rangle$.

Find $\kappa(t)$ of C. Describe the curve C.

Th. (13.2)

let u, v be vector fn's, c scalar, f real valued fn.
Then

1) $(c\,u(t))' = c\,u'(t)$.

2) $(u+v)' = u' + v'$

3) $(f u)' = f' u + f u'$

4) $(u \cdot v)' = (u \cdot v') + (u' \cdot v)$

5) $(u \times v)' = u \times v' + u' \times v$.