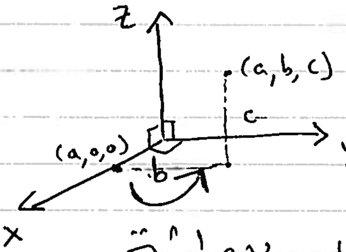
أنظمة الإحداثيات ثلاثية الأبعاد

12.1 Three – Dimensional Coordinate Systems

<u>Def^{n}</u> : The 3 – dim. rectangular coordinate system is

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (a,b,c) : a,b,c \in \mathbb{R} \}$



right hand rule

قاعدة اليد اليمنى

المحاور الإحداثية

The coordinate axis are ; x – axis, y – axis, z – axis.

المستويات ← الإحداثية

The coordinate planes are :

~~The first octant is $\{(a,b,c) : a>0, b>0, c>0\}$ الثماني الأول~~

xy – plane ($z=0$)

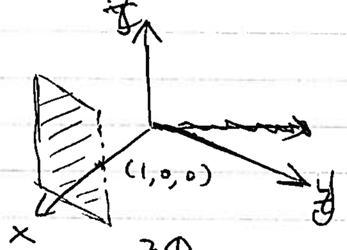
xz – plane ($y=0$)

yz – plane ($x=0$)

In $\mathbb{R}^3$, an eq$\underline{n}$ involving $x, y,$ and z is called a surface in $\mathbb{R}^3$

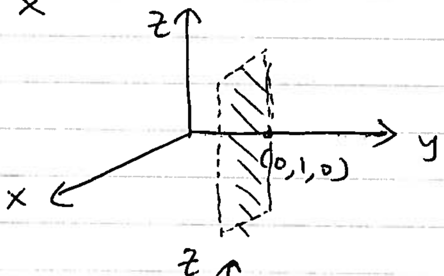
<u>Ex.</u> Describe and sketch the following surfaces in $\mathbb{R}^3$

1) $x = 1$



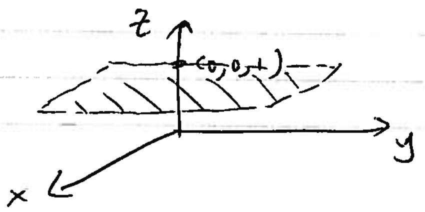
plane

2) $y = 1$



plane

3) $z = 1$

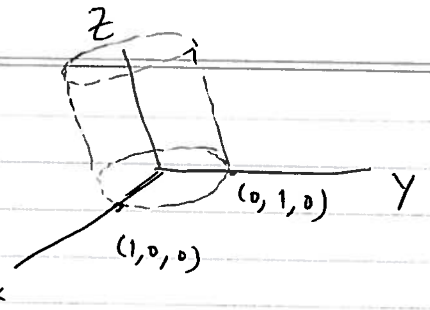


plane

4) $x^2 + y^2 = 1$

اسطوانة دائرية

circular cylinder



* The distance between $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$

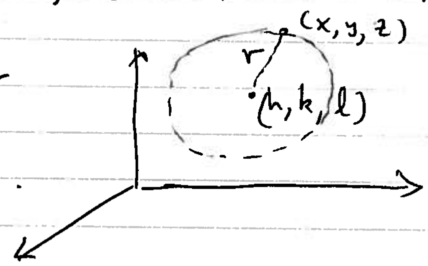
is : $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

* $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ is an equation

of a sphere (كرة) with center (مركز) (h, k, l) and radius (نصف قطر) r.

Pf : $\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r$

$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$.



Ex. Describe the following surfaces:

1) $2(x-2)^2 + 2y^2 + 2(z+3)^2 - 10 = 0$

$(x-2)^2 + y^2 + (z+3)^2 = 5$

sphere with center $(2, 0, -3)$ and radius $\sqrt{5}$.

2) $x^2 + y^2 + z^2 - 2x + 4y - 9 = 0$

$x^2 - 2x + y^2 + 4y + z^2 = 9$

$x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 = 9 + 1 + 4$

$(x-2)^2 + (y+2)^2 + z^2 = 14$

Sphere with center $(+2, -2, 0)$ and radius $\sqrt{14}$

3) $x^2+y^2+z^2-4x+8y+2z+21=0$

$(x-2)^2+(y+4)^2+(z+1)^2=-21+4+16+1$

$(x-2)^2+(y+4)^2+(z+1)^2=0$

the point $(2,-4,-1)$.

4) $x^2+y^2+z^2-4x+8y+2z+25=0$

$(x-2)^2+(y+4)^2+(z+1)^2=-25+4+16+1$

$(x-2)^2+(y+4)^2+(z+1)^2=-4$

$\therefore$ with no surface.

تقسم

Remark : In $\mathbb{R}^3$, the coordinate plane divide

أثمان

$\mathbb{R}^3$ into 8 octants.

The first octant is $\{(a,b,c) : a>0, b>0, c>0\}$.

Ex. Find the distance between $P_1=(-2,1,0)$, P_2 $(1,-1,2)$

$|P_1P_2| = \sqrt{(1+2)^2+(-1-1)^2+(2-0)^2}$

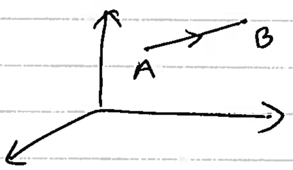
$= \sqrt{9+4+4} = \sqrt{17}$.

Exc Describe and sketch the surfaces

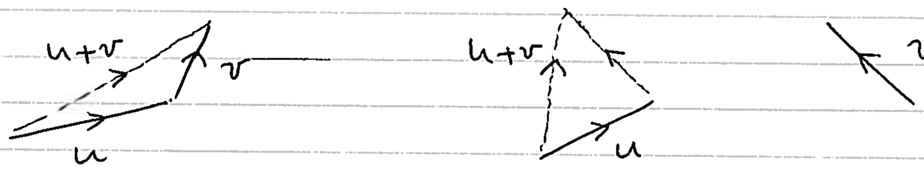
$x^2+z^2=1$ & $y^2+z^2=1$.

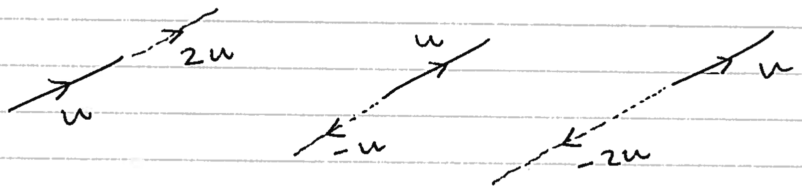
المتجهات

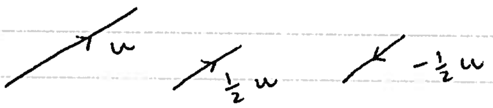
12.2 Vectors (3 - dimensional vectors)



$\overrightarrow{AB}$ is called a vector, where A is its initial point (الابتدائية) and B is its terminal point (النهائية)





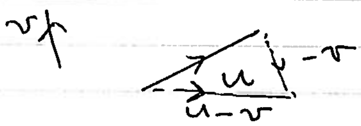


Def^n: The zero vector $\mathbf{0}$ is the vector whose initial point coincide with its terminal point.

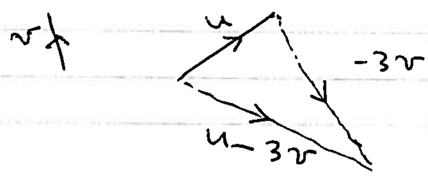
Def^n: $0u = \mathbf{0}$ & $c\mathbf{0} = \mathbf{0}$.

Remark: The length (الطول) of cu = $|c|$ (مضروبا في) times the length of u.
i.e $|cu| = |c||u|$.

Def^n: $u - v = u + (-v)$

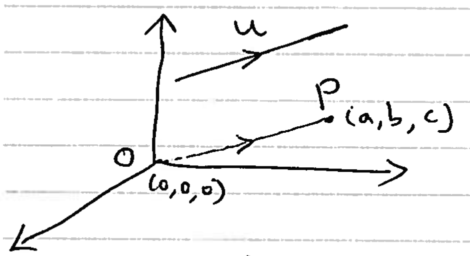


Ex.



المركبات

Components



متجه الوضع

$\overrightarrow{OP}$ is called the position vector of the point P.

الاحداثيات

We write $u = \langle a, b, c \rangle$ & the coordinates of P; a, b, c are called the components of u

مركبات

Remark: If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, then

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Remark: If $u = \langle a_1, a_2, a_3 \rangle$, then

$$|u| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Remark: If $u = \langle a_1, a_2, a_3 \rangle$, $v = \langle b_1, b_2, b_3 \rangle$, then

1) $u + v = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

2) $u - v = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

3) $cu = \langle ca_1, ca_2, ca_3 \rangle$.

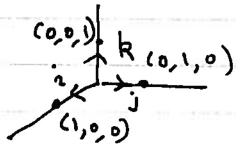
Remark: $0 = \langle 0,0,0 \rangle$ & $|0| = 0$

* the length of u is called also the magnitude of u denoted by $|u|$ or $\|u\|$.

Def^n: $i = \langle 1,0,0 \rangle$, $j = \langle 0,1,0 \rangle$, $k = \langle 0,0,1 \rangle$.

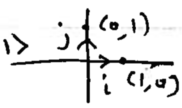
Remark: If $u = \langle a_1, a_2, a_3 \rangle$, then

$$u = a_1 i + a_2 j + a_3 k.$$



Remark: In 2-space, let $i = \langle 1,0 \rangle$, $j = \langle 0,1 \rangle$

If $u = \langle a_1, a_2 \rangle$, then $u = a_1 i + a_2 j$



Properties : Let u, v, w be vectors in $\mathbb{R}^3$,

a, b Scalars (اعداد), then.

1) $u + v = v + u$

2) $(u+v) + w = u + (v+w)$

3) $u + 0 = u$

4) $u + (-u) = 0$

5) $a(u+v) = au + av$

6) $(a+b)u = au + bu$

7) $1u = u$

8) $(ab)u = a(bu)$

* a vector with length 1 is called a unit vector.

Ex. Find a vector of length 8 and with opposite direction of the vector $u = 2i - j + 2k$

Soln. $u = \langle 2, -1, 2 \rangle$

$\|u\| = \sqrt{4+1+4} = 3.$

$$\left| \frac{u}{|u|} \right| = \frac{|u|}{|u|} = 1$$

Answer: $-8 \dfrac{u}{\|u\|} = -\dfrac{8}{3} \langle 2, -1, 2 \rangle$

$= \langle -\frac{16}{3}, 8/3, -\frac{16}{3} \rangle.$

$= -\frac{16}{3} i + 8/3\, j - \frac{16}{3} k$

Ex. $u = 2i - j + 3k$

$v = 3i + j - k$

Find $|2u - 3v|$

Soln $u = \langle 2, -1, 3 \rangle$, $v = \langle 3, 1, -1 \rangle$

$2u - 3v = \langle 4, -2, 6 \rangle - \langle 9, 3, -3 \rangle$

$= \langle -5, -5, 9 \rangle$

$|2u - 3v| = \sqrt{25 + 25 + 81} = \sqrt{131}$

Ex. If $\overrightarrow{AB} = 2i - j + 3k$, $A(-3, 1, -1)$, find B.

Soln $\overrightarrow{AB} = \langle 2, -1, 3 \rangle$.

let $B(x, y, z)$

$\langle x+3, y-1, z+1 \rangle = \langle 2, -1, 3 \rangle$

$x + 3 = 2 \rightarrow x = -1$

$y - 1 = -1 \rightarrow y = 0$

$z + 1 = 3 \rightarrow z = 2$

$\therefore B(-1, 0, 2)$.

Exc. let $u = 3i + j - k$.

Find 1) a unit vector in the opposite direction of u.

2) a vector with length 5 ~~...~~ in the direction of u.

الضرب النقطي .

12.3 The dot product

Def^n: If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the dot product of a and b is:

$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

Similarly, in 2-space, if $a = \langle a_1, a_2 \rangle$, $b = \langle b_1, b_2 \rangle$, then $a \cdot b = a_1 b_1 + a_2 b_2$.

Ex. $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = (2)(3) + (4)(-1) = 2$

$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1)(6) + (7)(2) + 4(-\frac{1}{2}) = 6$

$(i + 2j - 3k) \cdot (2j - k) = \langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle$

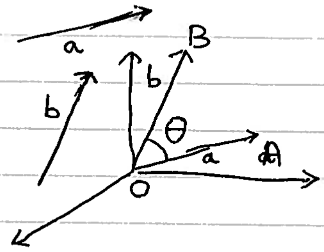
$= 1(0) + 2(2) + (-3)(-1) = 7$

$(2j - i) \cdot (2i + j) = \langle -1, 2 \rangle \cdot \langle 2, 1 \rangle = (-1)(2) + 2(1) = 0$

Properties: If a, b, and c are vectors in V_3 ($\mathbb{R}^3$),

and k is a scalar, then

1) $a \cdot a = |a|^2$
2) $a \cdot b = b \cdot a$
3) $a \cdot (b + c) = a \cdot b + a \cdot c$
4) $(ka) \cdot b = k(a \cdot b) = a \cdot (kb)$
5) $0 \cdot a = 0$



Def^n: The angle θ between two vectors a and b is the angle between $\vec{OA}$ and $\vec{OB}$ where $0 \le \theta \le \pi$.

Remark: If a is parallel to b, then $\theta = 0$ or $\theta = \pi$

$\theta = 0$ means that a and b have the same direction.

$\theta = \pi$ means that a and b have opposite directions

$\theta = \pi/2$ means that a and b are perpendicular (orthogonal)

The zero vector 0 is considered to be perpendicular to all vectors.

Th. $a \cdot b = |a||b| \cos\theta$. Thus $\cos\theta = \dfrac{a \cdot b}{|a||b|}$

Ex. If the vectors a and b have lengths 4 and 6, and the angle between them is $2\pi/3$, find $a \cdot b$.

Soln.: $a \cdot b = |a||b| \cos\theta$

$= 4(6) \cos 2\pi/3$

$= (24)(-\frac{1}{2}) = -12$

Ex. Find the angle between the vectors $a = \langle -1, -2, 1 \rangle$ and $b = \langle 2, 1, 1 \rangle$.

Soln. $|a| = \sqrt{1+4+1} = \sqrt{6}$ & $|b| = \sqrt{4+1+1} = \sqrt{6}$

$a \cdot b = -2 - 2 + 1 = -3$

So $\cos\theta = \dfrac{a \cdot b}{|a||b|} = \dfrac{-3}{\sqrt{6}\sqrt{6}} = \dfrac{-3}{6} = -\frac{1}{2}$.

Hence $\theta = \cos^{-1}(-\frac{1}{2}) = \pi - \frac{\pi}{3} = 2\pi/3$. (or $120°$)

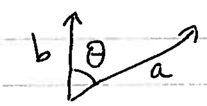
Remark: Two vectors a and b are orthogonal if and only if $a \cdot b = 0$. (both directions are correct).

Ex. Show that $2i + 2j - k$ is perpendicular to $5j - 4i + 2k$.

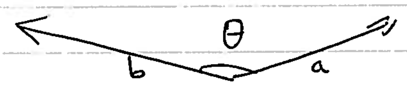
Soln.: $(2i + 2j - k) \cdot (5j - 4i + 2k) =$
$\langle 2, 2, -1 \rangle \cdot \langle -4, 5, 2 \rangle =$
$2(-4) + 2(5) + (-1)(2) = 0$

thus the vectors are perpendicular.

Remark:



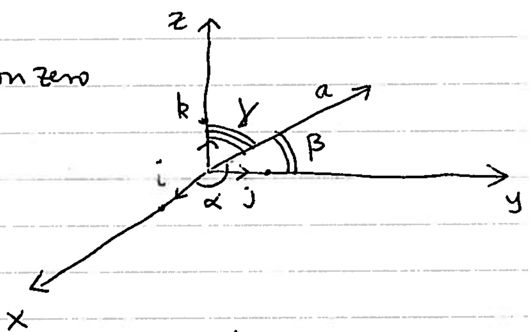
$0 < \theta < \frac{\pi}{2}$ (θ acute)
$\cos\theta > 0$
$a \cdot b > 0$
a and b point in the same general direction



$\frac{\pi}{2} < \theta < \pi$ (θ obtuse)
$\cos\theta < 0$
$a \cdot b < 0$
a and b point in generally opposite directions

Remark: If $\theta = 0$, then $a \cdot b = |a||b|$
If $\theta = \pi$, then $a \cdot b = -|a||b|$

Def^{n}: The direction angles of a nonzero vector a are α, β, and γ, where



α : angle between a and i
β : angle between a and j
γ : angle between a and k

Thus if $a = \langle a_1, a_2, a_3 \rangle$, then the cosines of the direction angles of a are $\cos\alpha = \frac{a \cdot i}{|a||i|} = \frac{a_1}{|a|}$

~~Cos~~ T

$$\cos\beta = \frac{a\cdot j}{|a||j|} = \frac{a_2}{|a|}, \quad \cos\gamma = \frac{a\cdot k}{|a||k|} = \frac{a_3}{|a|}$$

Also $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and

$$a = \langle a_1, a_2, a_3\rangle = \langle |a|\cos\alpha, |a|\cos\beta, |a|\cos\gamma\rangle$$

$$= |a|\langle \cos\alpha, \cos\beta, \cos\gamma\rangle \quad \text{thus}$$

$\langle \cos\alpha, \cos\beta, \cos\gamma\rangle = \dfrac{a}{|a|}$, that is, the direction cosines of a are the components of the unit vector in the direction of a

Ex. Find the direction angles of $a = \langle 1, -2, 3\rangle$

$|a| = \sqrt{1+4+9} = \sqrt{14}$.

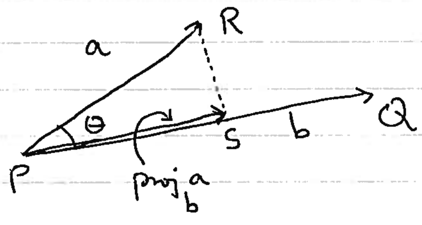
$\cos\alpha = \dfrac{1}{\sqrt{14}}$, $\cos\beta = \dfrac{-2}{\sqrt{14}}$, $\cos\gamma = \dfrac{3}{\sqrt{14}}$

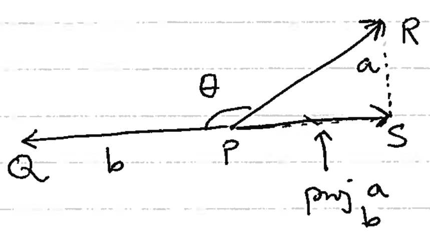
$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74^\circ$

$\beta = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right) = 180^\circ - \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 180^\circ - 58^\circ = 122^\circ$

$\gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$.

Projections:





Def^n: The vector projection of a onto b is:

$$\text{proj}_b\, a = \left(\frac{a \cdot b}{|b|}\right) \frac{b}{|b|} = \frac{a \cdot b}{|b|^2}\, b$$

Def^n: The scalar projection of a onto b (the component of a along b) is:

$$\text{comp}_b\, a = \frac{a \cdot b}{|b|} \qquad (= |a| \cos\theta)$$

Remark: $\text{proj}_b\, a = (\text{comp}_b\, a)$(the unit ~~projection~~ vector in the direction of b)

Ex. Find the scalar projection and vector projection of $a = \langle 1, 1, 2 \rangle$ onto $b = \langle -2, 3, 1 \rangle$.

Soln.
$$a \cdot b = (1)(-2) + 1(3) + 2(1) = 3$$

$$|b| = \sqrt{4+9+1} = \sqrt{14}.$$

$$\cancel{\text{proj}_b\, a} \; \cancel{=} \; \text{Comp}_b\, a = \frac{a \cdot b}{|b|} = \frac{3}{\sqrt{14}}$$

$$\text{proj}_b\, a = \cancel{a}\, (\text{comp}_b\, a)\left(\frac{b}{|b|}\right)$$
$$= \frac{3}{\sqrt{14}} \frac{\langle -2, 3, 1 \rangle}{\sqrt{14}}$$
$$= \frac{3}{14} \langle -2, 3, 1 \rangle = \left\langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

You can compute $\text{proj}_b\, a$ immediately:

$$\text{proj}_b\, a = \frac{a \cdot b}{|b|^2}\, b = \frac{3}{14} \langle -2, 3, 1 \rangle$$
$$= \left\langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

12.4 The cross product

Def^n: Let $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$. The cross product of a and b is:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k.$$

Ex. $a = \langle 1, 3, 4 \rangle$, $b = \langle 2, 7, -5 \rangle$. Then

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} k$$

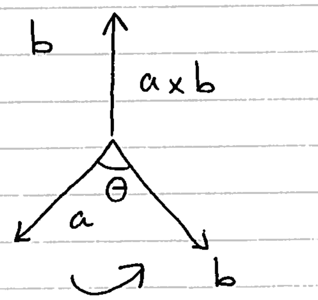
$$= (-15 - 28) i - (-5 - 8) j + (7 - 6) k$$

$$= (-43) i - (-13) j + k$$

$$= \langle -43, 13, 1 \rangle$$

Properties :

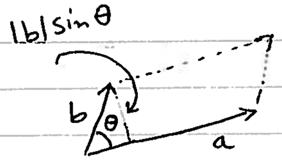
1) $a \times b$ is orthogonal to both a and b



2) $|a \times b| = |a||b| \sin\theta$

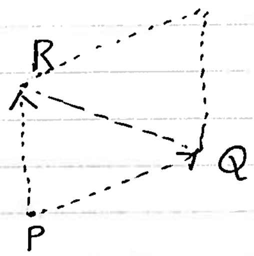
3) Two nonzero vectors a and b are parallel if and only if $a \times b = 0$

4) $|a \times b|$ is the area of the parallelogram determined by a and b



Ex. Find the area of the triangle with vertices

$P(1,4,6)$, $Q(-2,5,1)$, and $R(1,-1,1)$



$\overrightarrow{PQ} = \langle -3, 1, -5 \rangle$

$\overrightarrow{PR} = \langle 0, -5, -5 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -5 \\ 0 & -5 & -5 \end{vmatrix} = \langle -30, -15, 15 \rangle$$

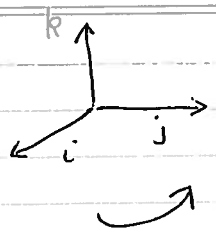
$$= -15 \langle 2, 1, -1 \rangle$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = 15\sqrt{(2)^2 + (1)^2 + (-1)^2} = 15\sqrt{6}$$

$\therefore$ the area of the parallelogram with adjacent sides PQ and PR is $|\overrightarrow{PQ} \times \overrightarrow{PR}| = 15\sqrt{6}$.

$\therefore$ The area of the triangle $PQR = \frac{1}{2}(15\sqrt{6}) = \frac{15\sqrt{6}}{2}$.

<u>Remark</u>:



$i \times j = k$

$j \times k = i$

$k \times i = j$

<u>Remark</u> Two nonzero vectors a and b are parallel if and only if $a = kb$ for some nonzero scalar k.

<u>Ex</u>. Let $a = \langle \frac{2}{3}, -\frac{1}{2}, \frac{1}{6} \rangle$

$b = \langle 4, -3, 1 \rangle$.

Show that a and b are parallel.

<u>Soln</u>. <u>Method I</u>:

Since $\underbrace{\langle 4, -3, 1 \rangle}_{b} = \underbrace{6 \langle \frac{2}{3}, -\frac{1}{2}, \frac{1}{6} \rangle}_{6a}$

it follows that a and b are parallel.

Method II: $a \times b = \begin{vmatrix} i & j & k \\ \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} \\ 4 & -3 & 1 \end{vmatrix}$

$= \langle 0, 0, 0 \rangle = 0$

So a and b are parallel.

Th.: Let a, b, c be vectors, k be a scalar. Then:

1) $0 \times a = 0$ & $a \times 0 = 0$ & $a \times a = 0$

2) $b \times a = -(a \times b)$, thus the cross product is not commutative

3) $k(a \times b) = (ka) \times b = a \times (kb)$.

4) $a \times (b + c) = (a \times b) + (a \times c)$

5) $(a + b) \times c = (a \times c) + (b \times c)$

6) $a \cdot (b \times c) = \cancel{(a \times b)} \, c \cdot (a \times b) = b \cdot (c \times a)$

$a \cdot (b \times c)$ is called the scalar triple product of the vectors a, b, and c

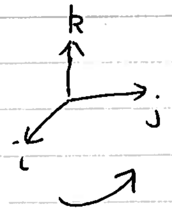
7) $a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$

$a \times (b \times c)$ is called the vector triple product of a, b, and c

Remark: The cross product is not associative

$i \times (i \times k) = i \times (-j) = -(i \times j) = -k$

$(i \times i) \times k = 0 \times k = 0$

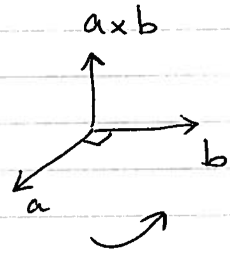


Thus $i \times (i \times k) \neq (i \times i) \times k$

Ex. Let a and b be orthogonal vectors, where

$|a| = 2$, $|b| = 3$. Find

$a \times (a \times b)$.



Soln.

Method I:

The direction of $a \times (a \times b)$ is $-b$

looks like i, j, k in the direction because a and b are orthogonal

$|a \times (a \times b)| = |a| \, |a \times b|$

$= (2)(2)(3) = 12$.

but $|b| = 3$.

So $a \times (a \times b) = -4b$.

Method II : Using property 7 of the previous th., we get $a\times(a\times b) = (a\cdot b)a - (a\cdot a)b$

$$= 0a - |a|^2 b$$

$$= 0 - 4b = -4b$$

Ex. If $a\cdot(b\times c) = -7$. Find

1) $(a\times c)\cdot b = b\cdot(a\times c)$
$= b\cdot(-c\times a)$
$= -(b\cdot(c\times c))$
$= -(-7)$ by property 6 of the previous theorem
$= 7$

2) $(a\times b)\cdot 2c = 2c\cdot(a\times b)$
$= 2(c\cdot(a\times b))$
$= 2(-7)$ property 6
$= -14$

3) $(b\times b)\cdot 2a = 0\cdot 2a = 0$

4) ~~$b\times$~~ $(2c\times b)\cdot 3a = [2(c\times b)]\cdot(3a)$

$= (2)(3)[(c\times b)\cdot a]$
$= 6[-(b\times c)\cdot a]$

$= -6[a\cdot(b\times c)]$
$= -6(-7) = 42.$

Th. If $a = \langle a_1, a_2, a_3 \rangle$, $b = \langle b_1, b_2, b_3 \rangle$, ~~then~~ $c = \langle c_1, c_2, c_3 \rangle$, then

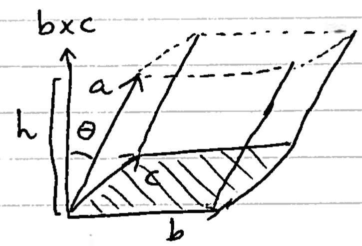
~~a~~

$$a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Remark: The volume of the parallelepiped determined by the vectors a, b, and c is:



$$V = |a \cdot (b \times c)|$$

because $|a \cdot (b \times c)| =$
$||a||b \times c| \cos\theta| =$
$|a||\cos\theta||b \times c| =$
(h)(the area of the parallelogram determined by b, c)
= the volume of the parallelepiped.

Remark: If $a \cdot (b \times c) = 0$, then a, b, and c lie in the same plane, that is, a, b, and c are coplanar. That is because the volume of the parallelepiped is 0

Ex. Use the scalar triple product to show that the vectors $a = \langle 1, 4, -7\rangle$, $b = \langle 2, -1, 4\rangle$, and $c = \langle 0, -9, 18\rangle$ are coplanar.

Soln: $a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$

$$= 9 \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 9\left(1\begin{vmatrix} -1 & 4 \\ -1 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + -7\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix}\right)$$

$$= 9\left((-2+4) - 4(4) - 7(-2)\right)$$

$$= 9(2 - 16 + 14) = 9(0) = 0$$

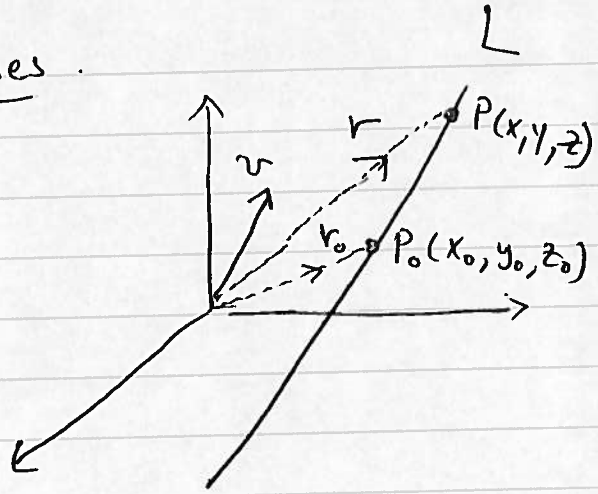
Thus a, b, and c are coplanar.

Exc. Find the volume of the parallelepiped ~~whose adjacent sides are~~ determined by $a = \langle 1, 2, 3\rangle$, $b = \langle -1, 1, 2\rangle$, $c = \langle 2, 3, -1\rangle$.

Exc. Given that $(b \times 2c) \cdot 4a = -32$,

find the volume of the parallelepiped determined by a, b, and c.

12.5 Equations of lines and planes.



Equations of lines:

Let L be a line in 3-space such that $P_0(x_0, y_0, z_0) \in L$ & $v = \langle a, b, c \rangle \parallel L$

Then $\overrightarrow{P_0P} \parallel v$ and thus

$\overrightarrow{P_0P} = tv$, where t is a scalar.

Also if $r = \langle x, y, z \rangle$ & $r_0 = \langle x_0, y_0, z_0 \rangle$ Then

$\boxed{r = r_0 + tv}$ This is called the vector equation of L

Also $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$

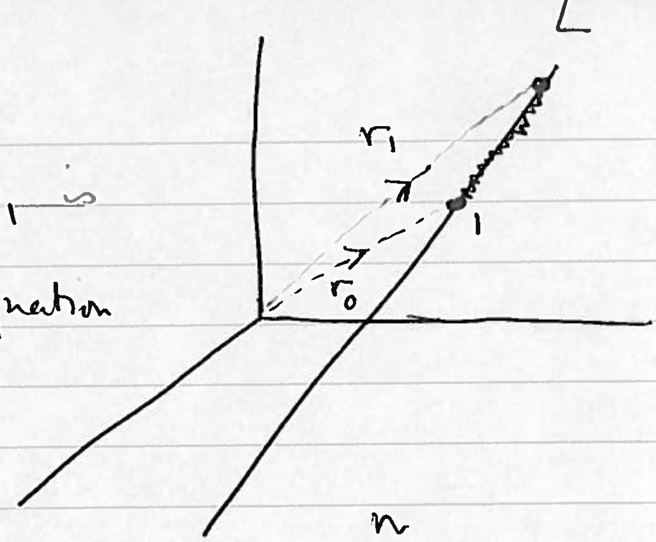
Thus
$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right]$$

These are called the parametric equations of L

Also $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ are called the symmetric equations of L

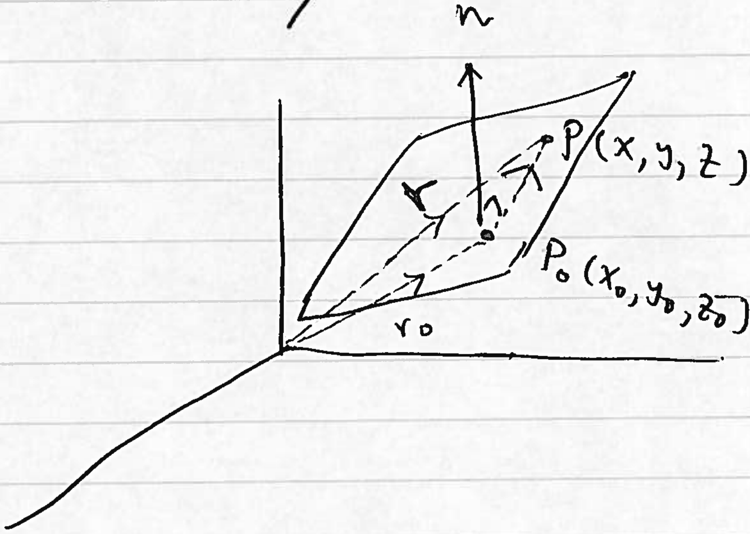
If $a = 0$, say, then the equations of L are

$x = x_0$, $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ This means that L lies in the ~~p~~ vertical plane $x = x_0$.



The line segment from r_0 to r_1 is given by the following vector equation

$$r(t) = (1-t)\,r_0 + t\,r_1, \quad 0 \le t \le 1$$



Planes : Let P be a plane such that

$P_0(x_0, y_0, z_0) \in P$ &

$n = \langle a, b, c\rangle \perp P$

If $r = \langle x, y, z\rangle$ &
$r_0 = \langle x_0, y_0, z_0\rangle$ Then

$\overrightarrow{P_0P} = r - r_0$ &

$\boxed{n \cdot (r - r_0) = 0}$ this is called the vector eqn. of P.

Also $\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$

Thus $\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$ This is called the scalar eqn. of P.

Remark : $ax + by + cz + d = 0$ is an equation of a plane having $n = \langle a, b, c\rangle$ as a normal.

* The distance from a point $P_0(x_0, y_0, z_0)$ to the plane

$P : ax + by + cz + d = 0$ is given by the following formula:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

* ~~By the angle between two planes, we mean the acute angle. Actually,~~ If θ is the acute angle between

P_1 & P_2 and $n_1 \perp P_1$, $n_2 \perp P_2$, then

$$\boxed{\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}}$$

* By skew lines L_1, L_2, we mean that L_1 & L_2 are neither parallel nor intersect.

* If θ is the acute angle between two intersecting lines L_1 & L_2,

$u_1 \parallel L_1$, $u_2 \parallel L_2$, then

$$\cos\theta = \frac{|u_1 \cdot u_2|}{|u_1||u_2|}$$

4)

Ex.1

1) Find a vector equation, parametric equations and symmetric equations for the line that passes through the point $P_0(5, 1, 3)$ and is parallel to the vector $v = i + 4j - 2k$

At what point does this line intersect the xy-plane.

2) $v = \langle 1, 4, -2 \rangle$, $(5, 1, 3) \in L$

$r = r_0 + t v$ vector eq$\underline{n}$.

$\langle x, y, z \rangle = \langle 5, 1, 3 \rangle + t \langle 1, 4, -2 \rangle$

param. eq$\underline{n}$. $x = 5 + t$, $y = 1 + 4t$, $z = 3 - 2t$.

Sym. eq$\underline{n}$. $x - 5 = \dfrac{y-1}{4} = \dfrac{z-3}{-2}$.

point of int. with xy-plane : $z = 0$

$z = 0 \longrightarrow 3 - 2t = 0 \longrightarrow t = 3/2$

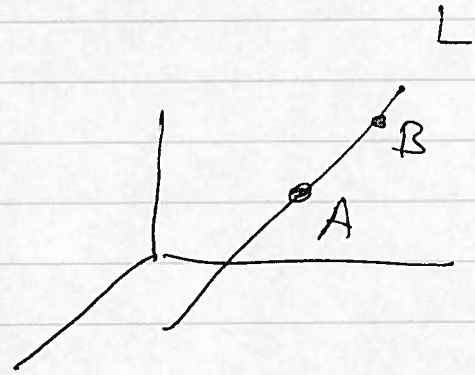
$x = 5 + t = 5 + 3/2 = 13/2$

$y = 1 + 4t = 1 + 4(3/2) = 1 + 6 = 7$

$\therefore$ the point of int. of L with xy-plane

is $\left(\frac{13}{2}, 7, 0\right)$.

Ex. 2 Find a vector eqn., parametric eqn's and Symmetric eqn's for the line passing through $A(2,4,-3)$, $B(3,-1,1)$.
At what point, this line intersect the xz-plane.



$\overrightarrow{AB} \parallel L$

$\overrightarrow{AB} = \langle 1, -5, 4 \rangle$

$A(2,4,-3) \in L$

vector eqn. $r = r_0 + t v$

$$\langle x, y, z \rangle = \langle 2, 4, -3 \rangle + t \langle 1, -5, 4 \rangle$$

param. eqn. $x = 2+t$, $y = 4-5t$, $z = -3+4t$

sym. eqn. $x - 2 = \dfrac{y-4}{-5} = \dfrac{z+3}{4}$

xz - plane : $y = 0$

$4 - 5t = 0 \longrightarrow t = 4/5$

$x = 2+t = 2 + 4/5 = \dfrac{14}{5}$

$z = -3 + 4t = -3 + \dfrac{16}{5} = \dfrac{1}{5}$

the pt. of int. with xz-plane is

$\left(\dfrac{14}{5}, 0, \dfrac{1}{5} \right)$.

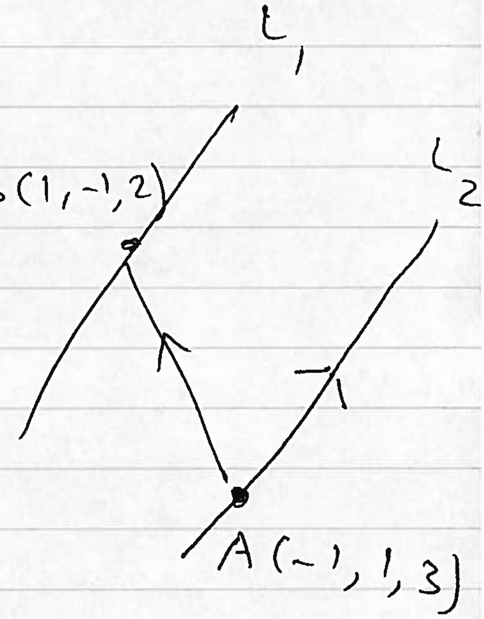
Ex.3 $L_1: x = t+1,\ y = 2t-1,\ z = 2-t$
$L_2: x = 2t-1,\ y = 4t+1,\ z = 3-2t.$

1) Show $L_1 \parallel L_2$ 2) Find the eqn. of the plane determined by L_1 & L_2.
3) Find the distance between L_1 & L_2.

1) $v_1 = \langle 1, 2, -1 \rangle \parallel L_1$
$v_2 = \langle 2, 4, -2 \rangle \parallel L_2.$

$v_1 \parallel v_2$ because $v_2 = 2v_1$. So $L_1 \parallel L_2$

2) $A(-1, 1, 3) \in L_2$



$B(1,-1,2) \in L_1$

$\vec{AB} = \langle 2, -2, -1 \rangle$

$L_2 \parallel \langle 2, 4, -2 \rangle = v_2$

$$\vec{AB} \times v_2 = \begin{vmatrix} i & j & k \\ 2 & -2 & -1 \\ 2 & 4 & -2 \end{vmatrix}$$

$= \langle 8, 2, 12 \rangle \perp$ the plane.

$n = \underset{\langle a,\ b,\ c \rangle}{\langle 8, 2, 12 \rangle} \perp P$

$A \underset{(x_0,\ y_0,\ z_0)}{(-1, 1, 3)} \in P$

eqn. of plane:

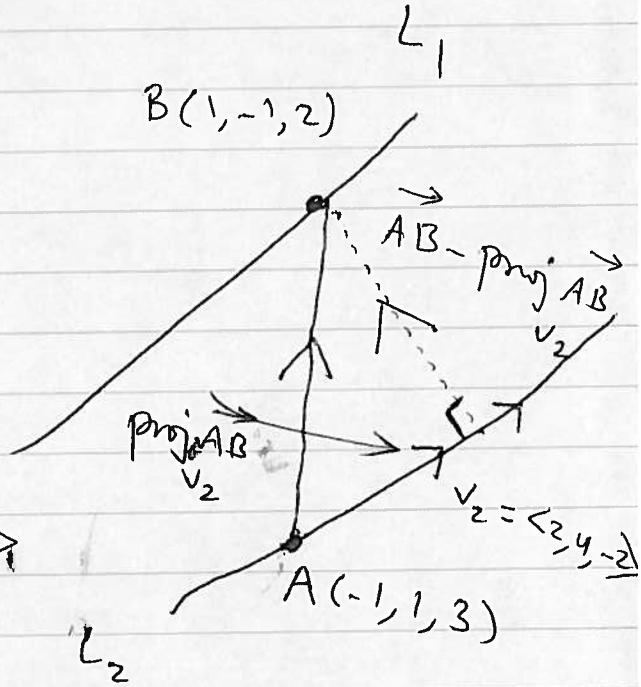
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$8(x+1) + 2(y-1) + 12(z-3) = 0$

$4(x+1) + y - 1 + 6(z-3) = 0$

$4x + y + 6z = 15$

3)



$$\text{proj}_{v_2} \vec{AB} = \frac{\vec{AB} \cdot v_2}{|v_2|^2} v_2$$

$$\vec{AB} = \langle 2, -2, -1 \rangle$$

$$v_2 = \langle 2, 4, -2 \rangle = 2 \langle 1, 2, -1 \rangle$$

$$\vec{AB} \cdot v_2 = 4 - 8 + 2 = -2$$

$$|v_2| = 2\sqrt{1+4+1} = 2\sqrt{6}$$

$$\text{proj}_{v_2} \vec{AB} = \frac{-2}{(4)(6)} \langle 2, 4, -2 \rangle = -\frac{1}{6} \langle 1, 2, -1 \rangle$$

$$= \langle -\frac{1}{6}, -\frac{1}{3}, \frac{1}{6} \rangle$$

$$\vec{AB} - \text{proj}_{v_2} \vec{AB} = \langle 2, -2, -1 \rangle - \langle -\frac{1}{6}, -\frac{1}{3}, \frac{1}{6} \rangle$$

$$= \langle \frac{13}{6}, \frac{-5}{3}, -\frac{7}{6} \rangle$$

The distance $= \left| \vec{AB} - \text{proj}_{v_2} \vec{AB} \right|$

$$= \sqrt{\left(\frac{13}{6}\right)^2 + \left(-\frac{5}{3}\right)^2 + \left(-\frac{7}{6}\right)^2}$$

Ex. 4 Let $L_1 : x-1 = \frac{y+2}{3} = \cancel{z}\ 4-z$ $\quad \frac{z-4}{-1}$

$L_2 : \frac{x}{2} = y-3 = \frac{z+3}{4}$

1) Show that L_1 & L_2 are skew

2) Find the distance between L_1 & L_2.

Soln 1) $v_1 = \langle 1, 3, -1 \rangle \parallel L_1$

$v_2 = \langle 2, 1, 4 \rangle \parallel L_2$ $\quad \langle 0,0,0 \rangle$

$$v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5 \rangle \neq 0$$

$\therefore v_1 \nparallel v_2 \quad \therefore L_1 \nparallel L_2.$

$L_1 : x = t+1, \ y = 3t-2 \quad z = 4-t$

$L_2 : x = 2s, \ y = s+3, \quad z = 4s-3$

$t+1 = 2s \Rightarrow t-2s = -1$ $\qquad t - 2s = -1$

$3t-2 = s+3 \qquad 3t - s = 5 \xrightarrow{\times -2} -6t + 2s = -10$

$-5t = -11 \Rightarrow t = \frac{11}{5}$

$t - 2s = -1 \Rightarrow \frac{11}{5} - 2s = -1 \longrightarrow 2s = \frac{11}{5} + 1 = \frac{16}{5}$

$\Rightarrow s = \frac{8}{5}$

$z = 4 - t = 4 - \frac{11}{5} = \frac{9}{5}$

$z = 4s - 3 = 4\left(\frac{8}{5}\right) - 3 = \frac{32}{5} - 3 = \frac{17}{5}$

$\frac{9}{5} \neq \frac{17}{5}$ $\therefore L_1$ & L_2 are not intersect.

$\therefore L_1$ & L_2 are skew.

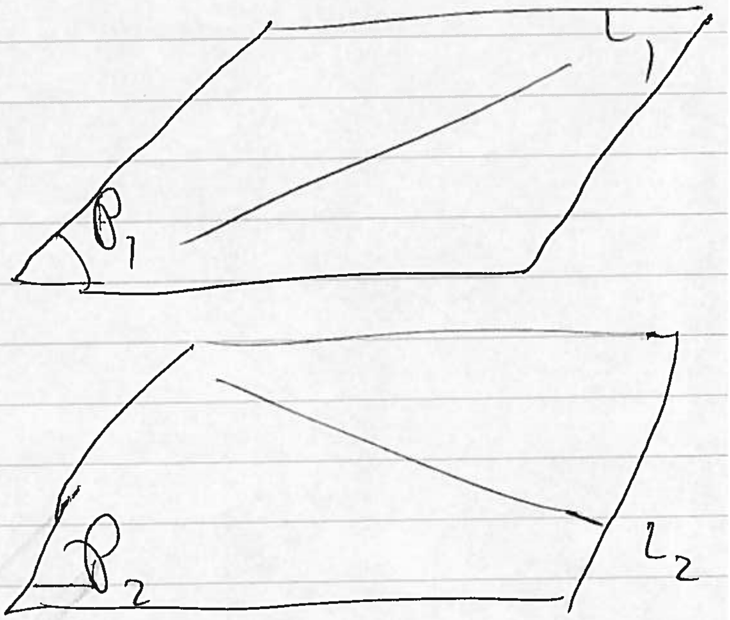
$$\langle a, b, c\rangle^{\underline{9}}$$

2) $v_1 \times v_2 = \langle 13, -6, -5\rangle$

$v_1 \times v_2 \perp P_2$

(x_0, y_0, z_0)
$(0, 3, -3) \in P_2$

equ. of P_2



$13(x-0) - 6(y-3) - 5(z+3) = 0$

$$\boxed{13x - 6y - 5z + 3 = 0}$$

The distance from L_1 to L_2 is
the distance between P_1 & P_2

choose $(1, -2, 4) \in P_1$

The distance from P_1 to P_2 is
the distance from $(1, -2, 4)$ to P_2 =

$$\frac{13(1) - 6(-2) - 5(4) + 3}{\sqrt{(13)^2 + (-6)^2 + (-5)^2}} = \frac{13 + 12 - 20 + 3}{\sqrt{169 + 36 + 25}}$$

$$= \frac{8}{\sqrt{230}}.$$

Ex. 5 Let $P_1 : x+y+z=1$ & $P_2 : x-2y+3z=1$

1) Find the acute angle between P_1 & P_2.
2) Find the parametric equ's for the line of intersection L of these two planes.

Soln 1) $n_1 = <1,1,1> \perp P_1$
$n_2 = <1,-2,3> \perp P_2$.

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|} = \frac{|1-2+3|}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}$$

$$\cancel{\cos\theta}\ \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right).$$

2) $x+y+z=1$ Let $x=0$

$y+z=1$

$x-2y+3z=1$ Let $x=0$

$-2y+3z=1$

$$y+z=1 \xrightarrow{\times 2} \cancel{2y}+2z=2$$
$$-2y+3z=1 \qquad \cancel{-2y}+3z=1$$

$5z=3 \Rightarrow z=3/5$

$y+z=1$

$y+3/5=1 \Rightarrow y=2/5$.

$\therefore A(0, 2/5, 3/5) \in L$
x_0, y_0, z_0

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$x = 1$

$x + y + z = 1 \longrightarrow 1 + y + z = 1 \Longrightarrow y + z = 0$

$x - 2y + 3z = 1$

$1 - 2y + 3z = 1 \Longrightarrow -2y + 3z = 0$

$y + z = 0$

$y = 0, \ z = 0$

$\therefore B(1, 0, 0) \in L.$

$$\overrightarrow{AB} = \underset{\langle a,\ b,\ c\rangle}{\langle 1, -\tfrac{2}{5}, \tfrac{-3}{5} \rangle} \ // \ L.$$

$\therefore$ param. eqn. for L are

$x = t$

$y = -\frac{2}{5}t + \frac{2}{5}$

$z = -\frac{3}{5}t + \frac{3}{5}$

~~Another method~~

Remark: Observe $n_1 \times n_2 \ // \ L$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle$$

$\langle 5, -2, -3 \rangle \ // \ L$

$A(0, \frac{2}{5}, \frac{3}{5}) \in L$

$\therefore$ param eqn. $x = 5t, \ y = \frac{2}{5} - 2t, \ z = \frac{3}{5} - 3t$

Ex.6 $L_1 : x = 2t-1,\ y = t+1,\ z = -t$
$L_2 : x = -t+1,\ y = 3t,\ z = t-7.$

Show $L_1 \perp L_2$.

$v_1 = \langle 2, 1, -1 \rangle \parallel L_1$

$v_2 = \langle -1, 3, 1 \rangle \parallel L_2$

$v_1 \cdot v_2 = -2 + 3 - 1 = 0$

$\therefore v_1 \perp v_2$

$\therefore L_1 \perp L_2.$

Ex.7 $P_1 : 2x + y - z + 1 = 0,\ P_2 : x - y + z = 5$

Show $P_1 \perp P_2$.

$n_1 = \langle 2, 1, -1 \rangle \perp P_1$

$n_2 = \langle 1, -1, 1 \rangle \perp P_2$

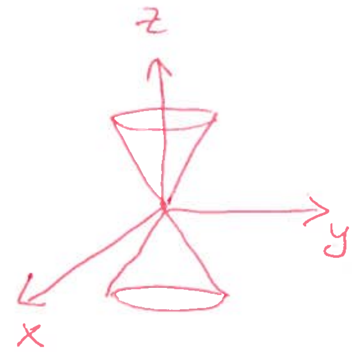
$n_1 \cdot n_2 = 2 - 1 - 1 = 0$

$\therefore n_1 \perp n_2$

$\therefore P_1 \perp P_2.$

12.6 Quadric surfaces

$\boxed{1}$ The cone $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



Ex. Sketch and identify the surface

$$x^2 + 4y^2 = 16z^2.$$

$$z^2 = \frac{x^2}{16} + \frac{y^2}{4} \quad (\text{cone}).$$

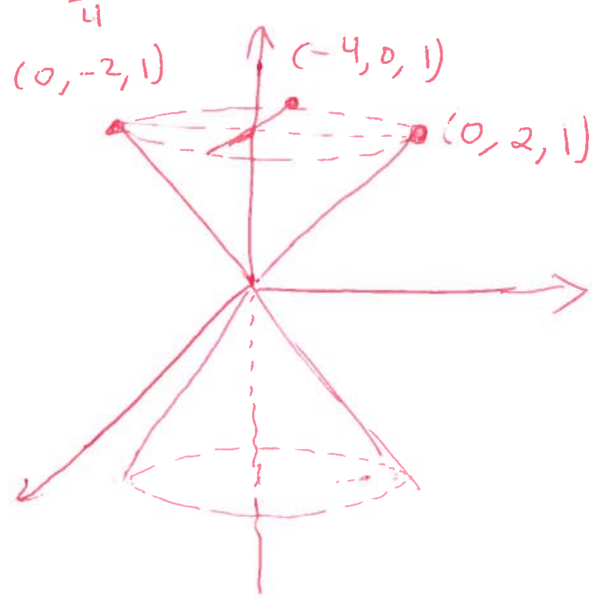
trace with $z = 1$ is : $\frac{x^2}{16} + \frac{y^2}{4} = 1$ ellipse

trace with $z = -1$ is : $\frac{x^2}{16} + \frac{y^2}{4} = 1$

observe that traces with xz-plane :

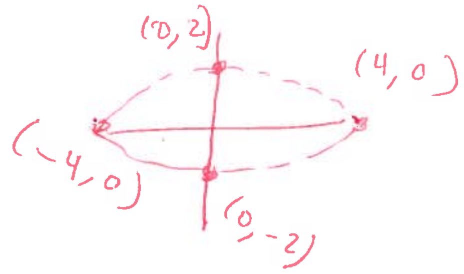
$y = 0 \Rightarrow z^2 = \frac{x^2}{16} \Rightarrow z = \pm \frac{x}{4}$ (lines)

elliptic cone



① symmetric about xy-plane because $z \to -z$ gives you the same eqn.

② sym. about xz-plane because $y \to -y$ gives the same eqn.



③ sym. about yz-plane because $x \to -x$ gives the same eqn.

$\boxed{2}$ elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (2)

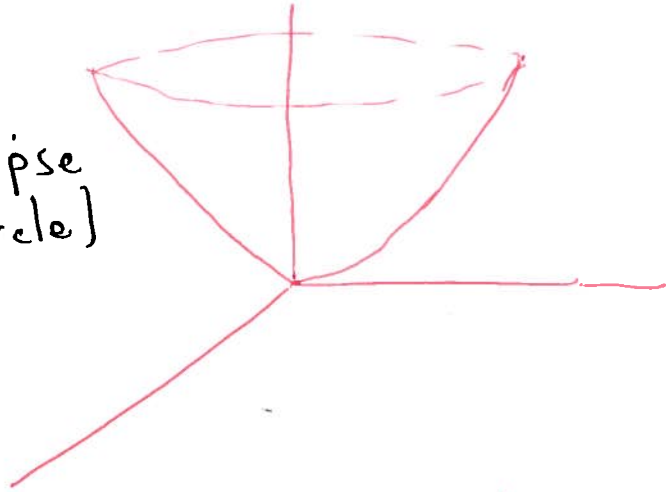
Ex: Sketch $4z = x^2 + y^2$ (sym. xz-plane
sym. yz-plane)

$$z = \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = \frac{x^2}{4} + \frac{y^2}{4}$$ elliptic paraboloid

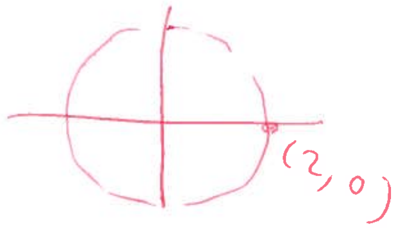
trace with $z = 1$ is :

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$ ellipse (circle)



trace with $\underbrace{\text{xz plane}}_{}$ ($y = 0$)

$$z = \frac{x^2}{4} + \frac{y^2}{4}$$



$$z = \frac{x^2}{4} + 0 \Rightarrow z = \frac{x^2}{4} \text{ (parabola)}$$

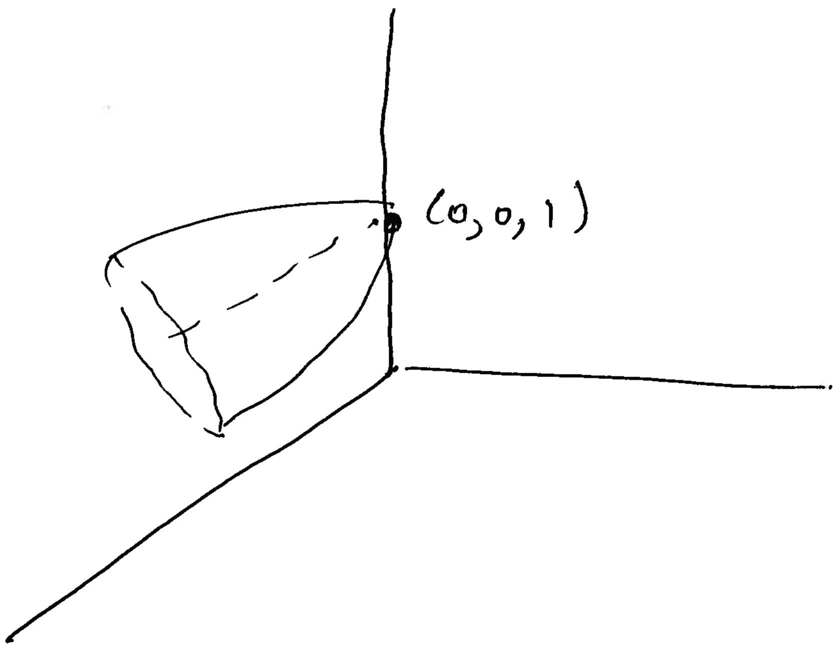
Ex. Identify & sketch

$$4x - z^2 + 2z - 1 - y^2 = 0$$

$$4x = z^2 - 2z + 1 + y^2$$

$$4x = (z-1)^2 + y^2$$

$$x = \frac{(z-1)^2}{4} + \frac{y^2}{4}$$ elliptic paraboloid (circular paraboloid)



$\boxed{3}$ ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$9x^2 + 4y^2 + 36z^2 = 36$

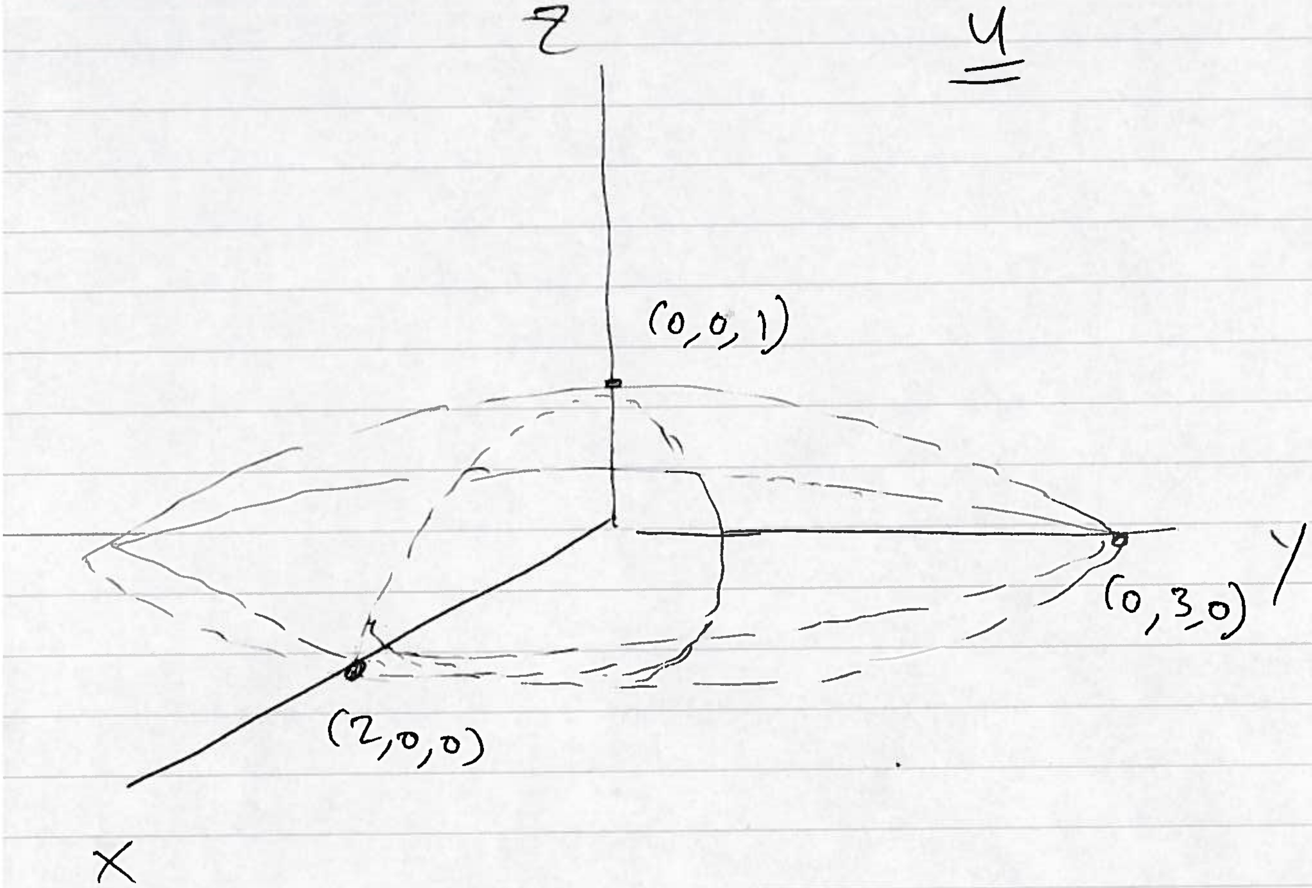
Ex. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

trace with $x = 0 \Rightarrow \frac{y^2}{9} + z^2 = 1$ (ellipse)

trace with $y = 0 \Rightarrow \frac{x^2}{4} + z^2 = 1$ (ellipse)

trace with $z = 0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$ (ellipse)

$\underline{\underline{y}}$



Exc. Identify and sketch the surface

$$y^2 + 4z^2 + 16x - 2y = -1$$