Special Theory of Relativity
Part II

Prof. Awni B. Hallak

Department of Physics
The Hashemite University
Zarqa 13115, JORDAN

Simultaneity

Why we are interested in simultaneous events and in their relative nature ?

It is all because recording the time of an event is done through another event occurring simultaneously.

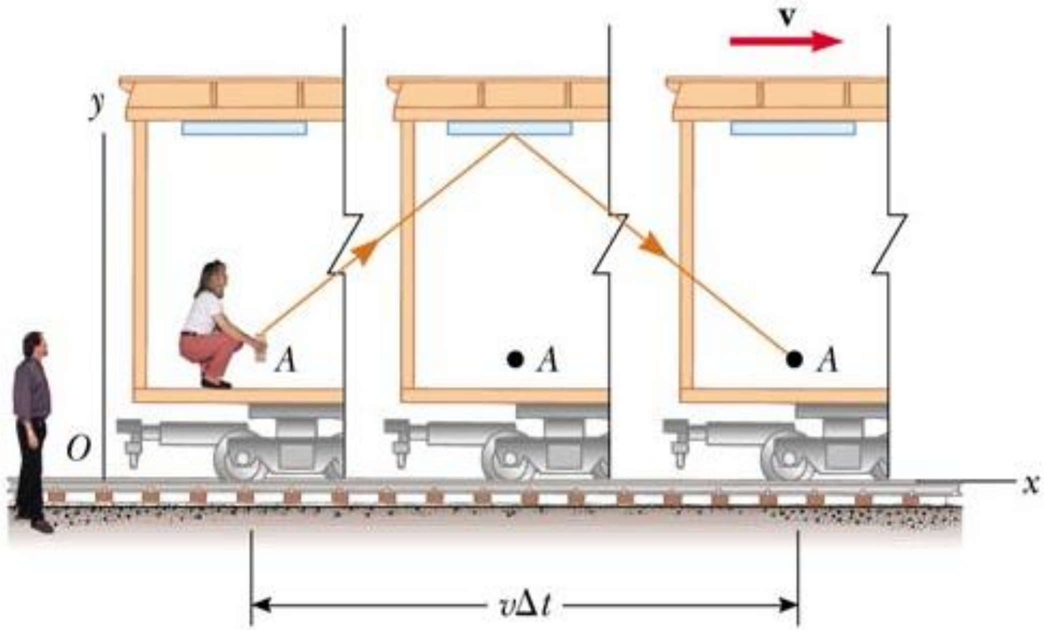
Example: A clock, at rest to an observer, is ticking 12:00 O'clock noon when an explosion takes place. Are these event simultaneous? The answer is yes. The observer records the explosion at 12:00 O'clock noon. However, if the observer and the clock are in a car moving with some speed, the events appear as not simultaneous.

Consequences of Relativity

Time dilation, length contraction

Time Dilation: setup

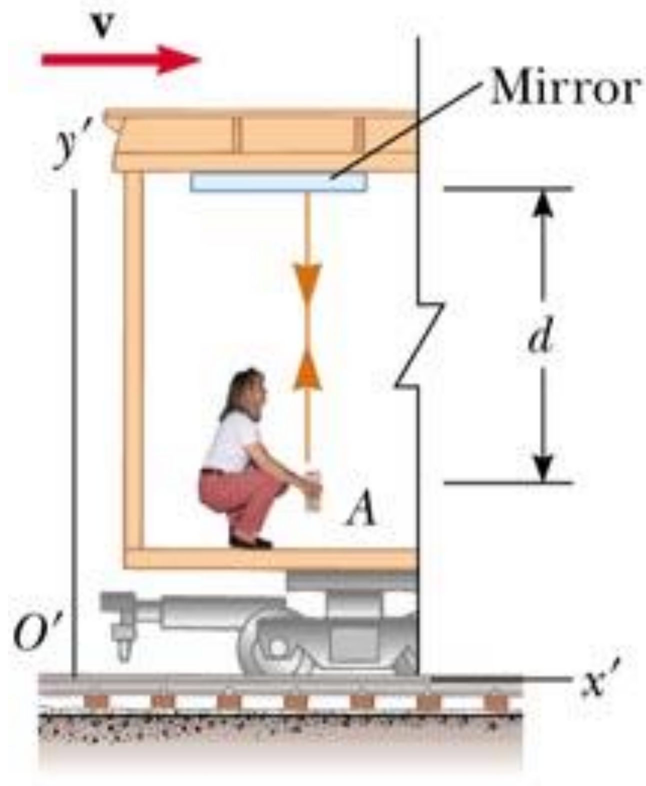
- **The concept of time interval is also not absolute**
- **To see this, imagine another boxcar experiment**
  - **Two observers, one in the car, another on the ground**



Time Dilation

Imagine an experiment:

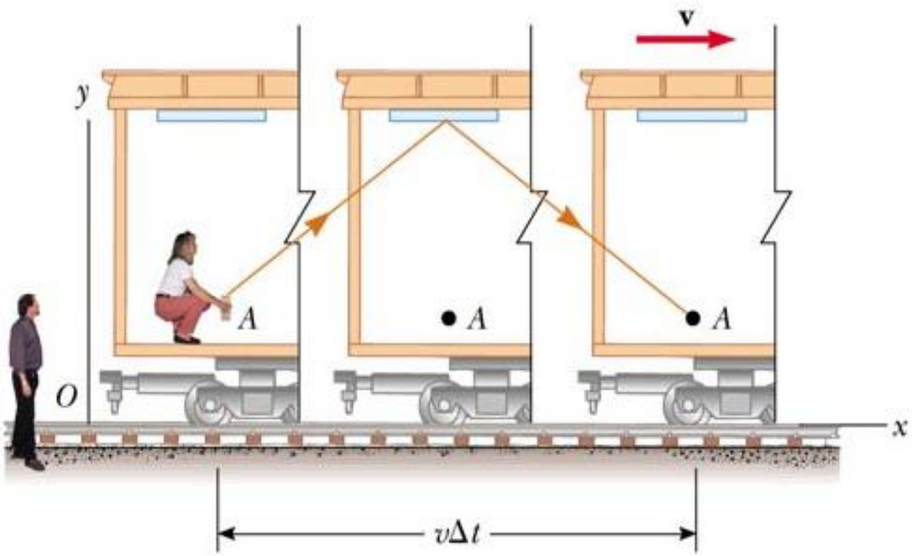
- **A mirror is fixed to the ceiling of a vehicle**
- **The vehicle is moving to the right with speed v**
- **An observer, O', at rest in this system holds a laser a distance d below the mirror**
- **The laser emits a pulse of light directed at the mirror (event 1) and the pulse arrives back after being reflected (event 2)**



Time Dilation, Moving Observer

- Observer O' carries a clock.
- She uses it to measure the time between the events.
  - She observes the events to occur within a time Δt_p.
  - Δt_p = distance/speed = $\left(\frac{2d}{c}\right)$.

Time Dilation, Stationary Observer



- Observer O is a stationary observer on the earth.
- He observes the mirror and O' to move with speed v.
- By the time the light from the laser reaches the mirror, the mirror has moved to the right.
- The light must travel farther with respect to O than with respect to O'.

Time Dilation, Observations

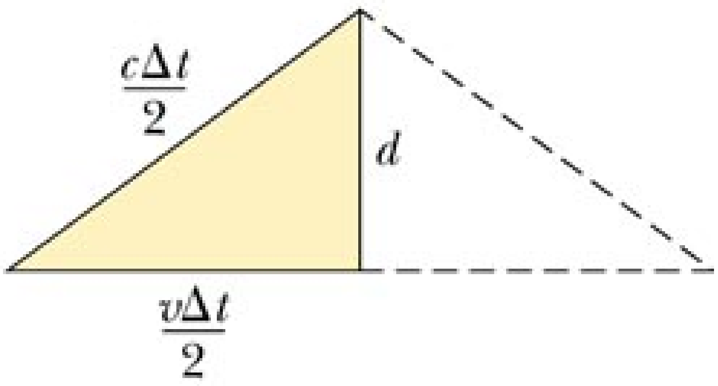
- Both observers must measure the speed of the light to be c.
- The light travels farther for O.
- The time interval, Δt, for O is longer than the time interval for O', Δt_p

Time Dilation, Time Comparisons

- $$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + d^2$$

  $$\frac{c^2}{4}(\Delta t)^2 - \frac{v^2}{4}(\Delta t)^2 = d^2$$

  $$(\Delta t)^2\left(1-\frac{v^2}{c^2}\right) = \left(\frac{2d}{c}\right)^2 = (\Delta t_p)^2$$



$$\Delta t = \frac{\Delta t_p}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \Delta t_p \text{ , where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

- **Observer O measures a longer time interval than observer O'.**

Time Dilation, Summary

- A clock moving at speed v past a stationary observer, runs more slowly (by a factor of $\frac{1}{\gamma}$) than an identical clock at rest with respect to the observer.
- The time interval Δt between two events measured by an observer moving with respect to a clock is longer than the time interval Δt_p between the same two events measured by an observer at rest with respect to the clock.

Identifying Proper Time

- **The time interval Δt_p is called the "proper time".**
  - **The proper time is the time interval between events as measured by an observer who sees the events occur at the same location.**
- **You must be able to correctly identify the observer who measures the "proper time" interval.**

Problem: Deep-Space Probe

A deep-space probe moves away from Earth with a speed of 0.80*c*. An antenna on the probe requires 3.0 s, probe time, to rotate through 1.0 revolution. How much time is required for 1.0 revolution according to an observer on Earth?

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Given:

$v = 0.8\ c$

$\Delta t_p = 3.0$ m/s

Find:

$\Delta t = ?$

Recall that the time on Earth will be longer then the proper time on the probe.

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Thus, numerically,

$$\Delta t = \frac{3.0\,s}{\sqrt{1-(0.8)^2}} = 5.0\,s$$

Alternate Views

- **The view of *O'* that *O* is really the one moving with speed v to the left and *O* clock is running more slowly is just as valid as *O* view that *O'* was moving.**

- **The principle of relativity requires that the views of the two observers in uniform relative motion must be equally valid and capable of being checked experimentally.**

Time Dilation – Generalization

- **All physical processes slow down relative to a clock when those processes occur in a frame moving with respect to the clock.**
  - **These processes can be chemical and biological as well as physical**

- **Time dilation is a very real phenomena that has been verified by various experiments.**

Lorentz Transformation (1)

- **Galilean Transformation equations are:**

$x' = x - vt$	$x = x' + vt$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = t$	$t = t'$

- **This transformation failed in many ways:**

  - Time is not absolute. (Time Dilation)

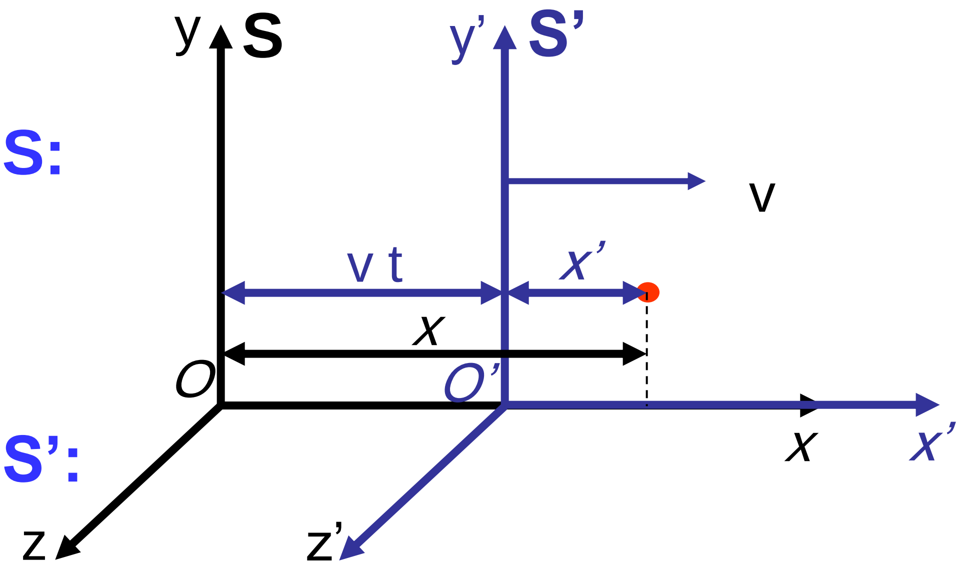
  - Length is not absolute. (Length Contraction)

  - Contradicts the fact that c is constant in all inertial frames.

  - And many others.

Lorentz Transformation (2)

Conclusion: Need a new set of transformation equations between (x’,u’,z’,t’) in S’ frame and (x,y,z,t) in S.



Spherical wavefront in S:

$$x^2 + y^2 + z^2 = c^2 t^2$$

Spherical wavefront in S’:

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

Lorentz Transformation (3)

Einstein made an intelligent guess. He suggested that:

$x' = \gamma\left(x - vt\right)$	$x = \gamma(x' + vt')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$ct' = x'$	$ct = x$

In what follows, we have a clear statement of the problem.

Lorentz Transformation (4)

Statement of the Problem:

- t' = t = 0 when S' and S have a common origin.
- A spherical light wave starts at t' = t = 0.
- According to Einstein's second Postulate, the speed of light c is constant in S and S'.
- The wavefronts observed in both systems must be spherical.
- Motion of S' is along x-axis $\Rightarrow$ x = c t , x' = c t'.
- x = k (x'- v t') , x' = k' (x – v t) , where k and k' are unknowns.
- k' = k according to Einstein's first Postulate $\Rightarrow$ k is the only unknown.
- y' = y and z' = z since motion is along the x-direction.

Lorentz Transformation (5)

Let

$$x' = k\left(x - vt\right) \qquad (1)$$

$$x = k'(x' + vt') \qquad (2)$$

By Einstein's first postulate (All laws of Physics must be the same in all inertial frames), we must have k' = k. i.e. k is the only unknown.

By Einstein's second postulate (The speed of light is constant in all inertial frames), we must have x = c t , x' = c t'.

Lorentz Transformation (6)

Substitute for x = c t and x' = c t', we have:

$$ct' = k(ct - vt) \qquad (1)$$

$$ct = k(ct' + vt') \qquad (2)$$

Or

$$t' = k\left(1 - \frac{v}{c}\right)t \qquad (3)$$

$$t = k\left(1 + \frac{v}{c}\right)t' \qquad (4)$$

Substitute from (4) into (3), we have:

Lorentz Transformation (7)

$$t' = k\left(1 - \frac{v}{c}\right)\left[k\left(1 + \frac{v}{c}\right)t'\right]$$

$$1 = k^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow$$

$$k = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

Equations (1) and (2) now become

$$x' = \gamma(x - vt) \quad (1)$$

$$x = \gamma(x' + vt') \quad (2)$$

Lorentz Transformation (8)

To calculate the "times" equations, we substitute from (1) into (2). We have:

$$x = \gamma\left[\gamma(x - vt) + vt'\right] = \gamma^2(x - vt) + \gamma v t'$$

$$\frac{x}{\gamma v} = \frac{\gamma^2 x}{\gamma v} - \gamma t + t'$$

$$t' = \left(\frac{x}{\gamma v}\right)\left(1 - \gamma^2\right) + \gamma t = \left(\frac{x}{\gamma v}\right)\left(1 - \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\right) + \gamma t$$

$$= \left(\frac{x}{\gamma v}\right)\left(1 - \frac{c^2}{c^2 - v^2}\right) + \gamma t = \left(\frac{x}{\gamma v}\right)\left(\frac{-v^2}{c^2 - v^2}\right) + \gamma t$$

Lorentz Transformation (9)

$$t' = \gamma\left[t - \left(\frac{xv}{c^2}\right)\right] \qquad (5)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We can also find the expression for t to be:

$$t = \gamma\left[t' + \left(\frac{x'v}{c^2}\right)\right] \qquad (6)$$

Lorentz Transformation (10)

Summary of Lorentz Equations:

$x' = \gamma\left(x - vt\right)$	$x = \gamma\left(x' + vt'\right)$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = \gamma\left(t - \frac{xv}{c^2}\right)$	$t = \gamma\left(t' + \frac{x'v}{c^2}\right)$

Some Remarks

1) If v << c, i.e., $\beta \approx 0$ and $\gamma \approx 1$, we see these equations reduce to the familiar Galilean transformation.

2) Space and time are now not separated.

3) For non-imaginary transformations, the frame velocity cannot exceed *c*.

Some consequences of the above equations follow.

Some Consequences of Lorentz Transformation

- **Time Dilation is predicted from Lorentz Transformation equations.**
- **Synchronization of clocks in different inertial frames is necessary.**
- **Length Contraction is also predicted from Lorentz Transformation equations.**
- **Lorentz Transformation of Velocities in different inertial frames is derivable from Lorentz Transformation equations.**
- **Lorentz Transformation equations are in agreement with Einstein's Second Postulate.**

1. Time Dilation (Again)

Consider the time interval between two events as reported in the two inertial frames S and S'. The time interval T in the S frame (at rest) and T' in the S' frame (moving with constant velocity v) are:

$$T = t_2 - t_1 \quad , \quad T' = t'_2 - t'_1$$

Where the relations between the reported times are given by Lorentz Transformation equations:

$$t_1 = \gamma\left(t'_1 + \frac{v\,x'}{c^2}\right) \quad , \quad t_2 = \gamma\left(t'_2 + \frac{v\,x'}{c^2}\right)$$

$$t_2 - t_1 = \gamma\left(t'_2 - t'_1\right) \quad \Rightarrow \quad T = \gamma\, T_o$$

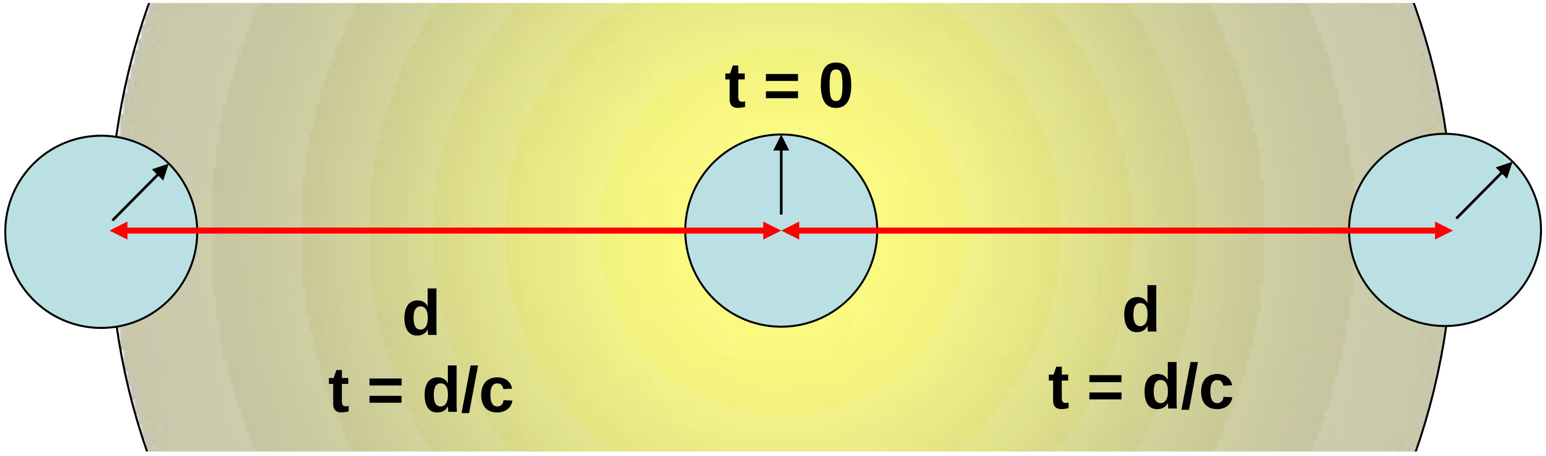
2. Synchronization of Clocks

- **Step 1: Place observers with clocks throughout a given system.**
- **Step 2: In that system bring all the clocks together at one location.**
- **Step 3: Compare the clock readings.**
- **If all of the clocks agree, then the clocks are said to be synchronized.**

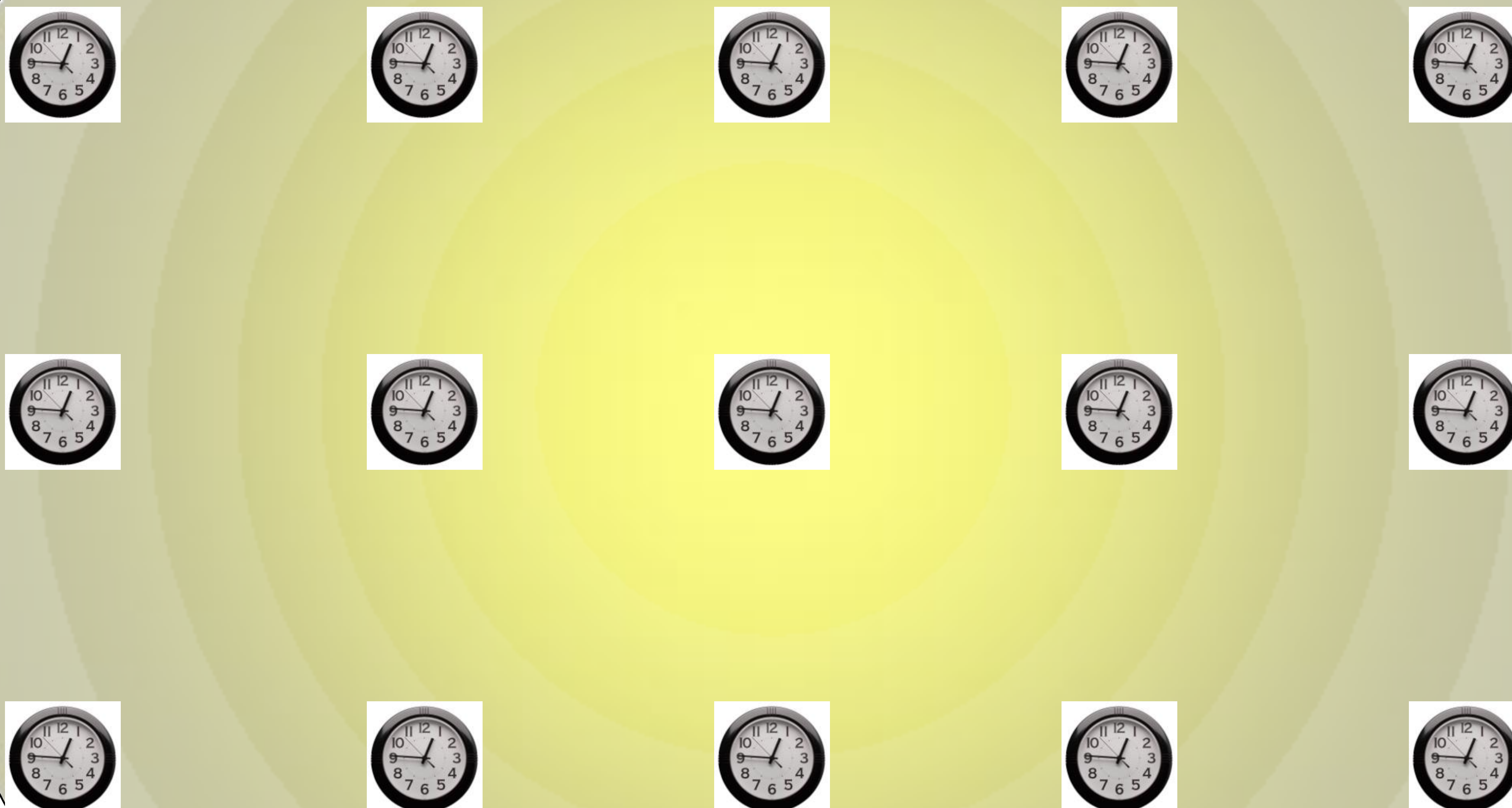
A method to synchronize Clocks

- One way is to have one clock at the origin set to t = 0 and advance each clock by a time (d/c) where d is the distance of the clock from the origin.

  - Allow each of these clocks to begin timing when a light signal arrives from the origin.



Synchronization of clocks



The Twin Paradox – Statement of the Problem

- **A thought experiment involving two twins named Speedo and Goslo.**
- **Speedo travels to Planet X, 20 light years from earth.**
  - **His ship travels at 0.95c.**
  - **After reaching planet X, he immediately returns to earth at the same speed.**
- **When Speedo returns, he has aged 13 years, but Goslo has aged 42 years.**

The Twins' Perspectives

- **Goslo's perspective is that he was at rest while Speedo went on the journey.**
- **Speedo thinks he was at rest and Goslo and the earth raced away from him on a 6.5 year journey and then headed back toward him for another 6.5 years.**
- **The paradox – which twin is the traveler and which is really older?**

The Twin Paradox – The Resolution

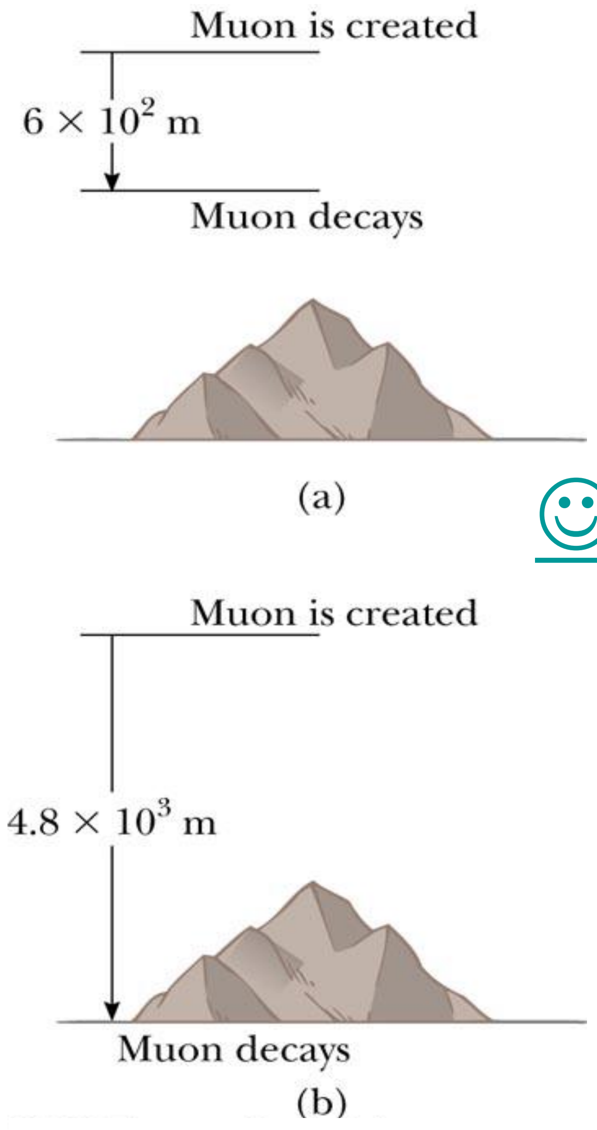
- **Relativity applies to reference frames moving at uniform speeds.**
- **The trip in this thought experiment is not symmetrical since Speedo must experience a series of accelerations during the journey.**
- **Therefore, Goslo can apply the time dilation formula with a proper time of 42 years.**

$$T = T_p\sqrt{1-\left(\frac{v}{c}\right)^2} = 42\sqrt{1-(0.95)^2} = 13.1$$

  - **This gives a time for Speedo of 13.1 years and this agrees with the earlier result.**
- **There is no true paradox since Speedo is not in an inertial frame.**

Time Dilation Verification – Muon Decays

- **Muons are unstable particles that have the same charge as an electron, but a mass 207 times more than an electron.**
- **Muons have a half-life of Δt_p = 2.2 μs when measured in a reference frame at rest with respect to them (Figure a).**
- **Relative to an observer on earth, muons should have a lifetime of $\gamma\ \Delta t_p$ (Fugure b).**
- **A CERN experiment measured lifetimes in agreement with the predictions of relativity.**



3. Length Contraction

- **The measured distance between two points depends on the frame of reference of the observer**

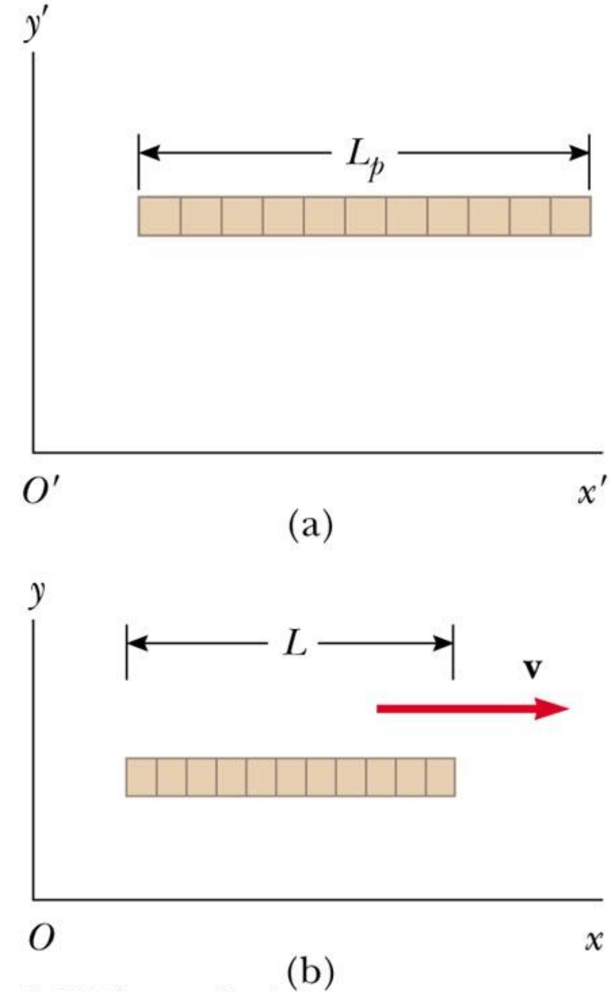
- **The *proper length*, L_p, of an object is the length of the object measured by someone at rest relative to the object**

- **The length of an object measured in a reference frame that is moving with respect to the object is always less than the proper length**
  - **This effect is known as *length contraction***

Length Contraction – Equation

- $$L = \frac{L_P}{\gamma} = L_P\sqrt{1 - \frac{v^2}{c^2}}$$

- **Length contraction takes place only along the direction of motion.**



Length Contraction – Derivation(1)

Here we derive Length Contraction from Lorentz Transformation equations:

The proper length L_p of a rod as measured in the moving frame S' is given by:

$$L_p = x_2' - x_1'$$

The length L as measured in the stationary frame S is given by:

$$L = x_2 - x_1$$

Length Contraction – Derivation(2)

In S', measurements of the two position coordinates that specify the length L_p are given by Lorentz equation:

$$x_2' = \gamma(x_2 - v t_2) \ , \ \ x_1' = \gamma(x_1 - v t_1)$$

$\Rightarrow$

$$L_p = x_2' - x_1' = \gamma(x_2 - v t_2) - \gamma(x_1 - v t_1)$$

In S, $t_2 = t_1$ since x_2 and x_1 are measured simultaneously.

Length Contraction – Derivation(3)

Expanding the brackets we get:

$$L_p = x_2' - x_1' = \gamma x_2 - \gamma v t_1 - \gamma x_1 + \gamma v t_1$$

$\Rightarrow$

$$L_p = \gamma(x_2 - x_1) = \gamma L$$

or

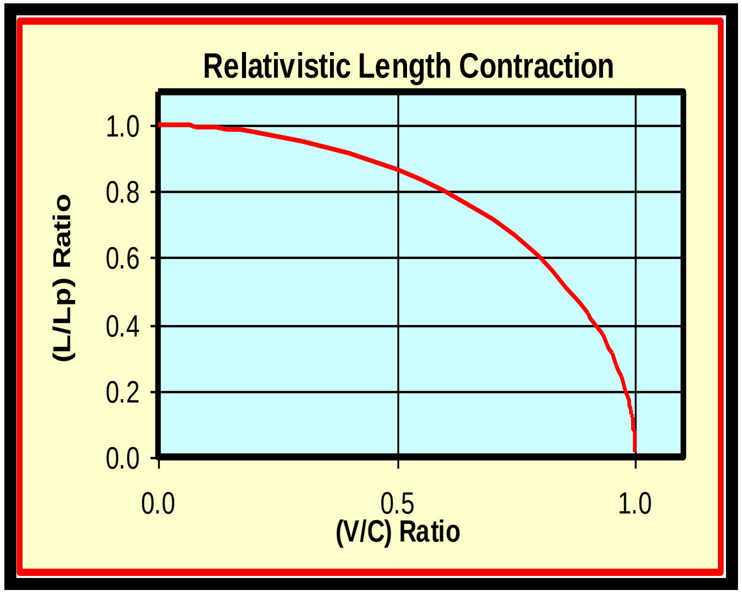
$$L = \frac{L_p}{\gamma}$$

But

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow L = L_p \sqrt{1 - \frac{v^2}{c^2}}$$

Length Contraction – Illustration

v/c	L/L_p	v/c	L/L_p
0.1000	0.9950	0.9600	0.2800
0.4000	0.9165	0.9900	0.1411
0.6000	0.8000	0.9920	0.1262
0.8000	0.6000	0.9940	0.1094
0.9000	0.4359	0.9950	0.0999
0.9100	0.4146	0.9960	0.0894
0.9200	0.3919	0.9970	0.0774
0.9300	0.3676	0.9980	0.0632
0.9400	0.3412	0.9990	0.0447
0.9500	0.3122	0.9999	0.0141



4. Equivalence of L.T. and G.T. in the Non-relativistic Limit

When v<<c, we have from L.T. first equation:

$$\mathbf{x}' = \gamma(\mathbf{x} - \mathbf{v}\,\mathbf{t})$$

where $$\gamma = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{\mathbf{c}^2}}}$$

When v<<c, $\gamma \rightarrow 1$

We get agreement with G.T. equation.

However, because of Time Dilation and Length Contraction, Time is NOT absolute. Here we have an Illustration.

4. Equivalence of L.T. and G.T. in the Non-relativistic Limit

Consider the motion of the origin of O'. t'=t=0 at the start of the motion.

After time t in O, O' has moved a distance x=vt.

Substitute for x in the time equation in L.T.

$$t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{xv}{c^2}\right) = \frac{\left(t - \frac{v(vt)}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = t\,\frac{\left(1 - \frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= t\sqrt{1-\frac{v^2}{c^2}}\ . \text{ When } v \ll c \Rightarrow t' = t$$

5. Lorentz Velocity Transformation L.V.T.

From L.T. equations, we have:

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

Let $\frac{dx'}{dt'} = u'_x \quad , \quad \frac{dx}{dt} = u_x$

$\Rightarrow$

$$dx' = \gamma(dx - v\,dt)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$\Rightarrow \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\left(\frac{dx}{dt}\right) - v}{1 - \frac{v}{c^2}\left(\frac{dx}{dt}\right)}$$

or

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}$$

5. Lorentz Velocity Transformation L.V.T.

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2} u_x} \quad \text{——— (1)}$$

Similarly, if the object under study has velocity components along the y and z axes, the components as measured by an observer in S' are:

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{——— (2)}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{——— (3)}$$

5. Lorentz Velocity Transformation Summary of Equations

$$u'_x = \frac{u_x - v}{\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{———} (1) \text{———} \quad u_x = \frac{u'_x + v}{\left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{———} (2) \text{———} \quad u_y = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2} u_x\right)} \quad \text{———} (3) \text{———} \quad u_z = \frac{u'_z}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)}$$

6. Constancy of the speed of light (1)

If u_x = c then u'_x = c

From eq. (1), put u_x= c

$\Rightarrow$

$$u'_x = \frac{c - v}{1 - \frac{v}{c^2}c} = c$$

Also

If u'_x = c then u_x = c

From eq. (1), put u'_x= c

$\Rightarrow$

$$u_x = \frac{c + v}{1 + \frac{v}{c^2}c} = c$$

6. Constancy of the speed of light (2)

The speed of an object can never exceed the velocity of light c.

Let $u'_x = k\,c$, where k is a positive constant. Also let S' travels at speed $v = k\,c$ relative to S.

Now we find u_x:

$$u_x = \frac{kc + kc}{1 + \frac{kc}{c^2}kc} = c\left(\frac{2k}{1 + k^2}\right)$$

Let us evaluate the maximum value of k.

6. Constancy of the speed of light (3)

Let

$$A = \left(\frac{2k}{1+k^2} \right)$$

A is a maximum when $\dfrac{dA}{dk} = 0$

$$\frac{dA}{dk} = \left(\frac{(1+k^2)(2) - 2k(2k)}{(1+k^2)^2} \right) = 0$$

$$1+k^2 - 2k^2 = 0 \quad \Rightarrow k = +1$$

Substitute the above result in A, you get:

$$A(\text{maximum}) = 1 \quad \Rightarrow \quad u_x(\text{maximum}) = c$$

6. Constancy of the velocity of light (4)

The speed of light c is the same in all directions in all inertial frames.

In slide 123, we showed that the speed of light propagating along x'-axis and the frame S' moves with velocity v along the same axis.

Consider next the case of observers moving along O'y' at right angle to the direction of propagation of light which is O'x'.

$$\Rightarrow \boxed{u'_x = 0 \quad , \quad u'_y = c \quad , \quad u'_z = 0}$$

6. Constancy of the speed of light (5)

$$u'_x = 0 \quad , \quad u'_y = c \quad , \quad u'_z = 0$$

Substitute in the U_x equation in L.T. $\Rightarrow$

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x} = v \quad , \quad u_y = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2} u'_x\right)} = \frac{c}{\gamma}$$

$$u_z = \frac{u'_z}{1 - \frac{v}{c^2} u'_x} = 0 \quad \Rightarrow \quad u = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

$\Rightarrow$

$$u = \sqrt{c^2 + \frac{c^2}{\gamma^2}} = \sqrt{c^2 + c^2\left(1 - \frac{v^2}{c^2}\right)} = c$$

6. Constancy of the speed of light (6)

$$c + c = c!$$

If we substitute $u'_x = c$ and $v = c$ in the u_x equation in L.T. $\Rightarrow$

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x} = \frac{c + c}{1 + \frac{c}{c^2} c} = c$$

Conclusion:

The relative velocity of two objects or two frames or an object in a frame can not exceed c.