# Complete set of orthogonal functions

## Discrete set of vectors:

**The two vectors** $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ **and** $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ **are orthogonal if** $\vec{A} \bullet \vec{B} = 0$ **or** $\sum_{i=1}^{3} A_i B_i = 0$**.**

## Continuous set of functions on an interval (a, b):

**a) The two continuous functions A*(x)* and B *(x)* are orthogonal on the interval (a, b) if** $\int_a^b A(x)B(x)dx = 0$**.**

**b) The two complex functions A *(x)* and B *(x)* are orthogonal on the interval (a, b) if** $\int_a^b A^*(x)B(x)dx = 0$**, where A*(*x*) is the complex conjugate of A (*x*).**

**c) For a whole set of functions $A_n$ (*x*) (where n= 1, 2, 3,) and on the interval (a, b)**

$$\int_a^b A_n^*(x)A_m(x)dx = \begin{cases} 0 & if \quad m \neq n \\ const.t \neq 0 & if \quad m = n \end{cases},$$

$A_n$ (*x*) is called a set of orthogonal functions**.**

## Examples:

***i*)** $$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \begin{cases} 0 & if \quad m \neq n \\ \pi & if \quad m = n \neq 0 \end{cases},$$

**where sin *nx* is a set of orthogonal functions on the interval (-π, π).**

***Similarly***

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0 & if \quad m \neq n \\ \pi & if \quad m = n \neq 0 \end{cases}$$

***ii)*** $\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$ ***for any n and m***

***iii)*** $\int_{-\pi}^{\pi} (e^{inx})^* e^{imx} dx = \begin{cases} 0 & if \quad m \neq n \\ 2\pi & if \quad m = n \neq 0 \end{cases}$

***vi)*** $\int_{-1}^{1} P_\ell(x) P_m(x) dx = 0$ ***unless.*** $\ell = m$

***[Try to prove this; also solve problems (2, 5) of section 6].***

***v)*** $\int_{-1}^{1} P_\ell(x) \bullet$ ***(any polynomial of degree <*** $\ell$ ***) dx = 0.***

***[Solve problems (6 of section 6) & (4, 5 and 6 of section 7)].***

**Conclusion*:* *The vector*** $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ **in 3-D is expanded in terms of unit vectors** $\hat{i}, \hat{j}$ **and .**$\hat{k}$ **Then** $\hat{i}, \hat{j}$ **and** $\hat{k}$ **form complete set of orthogonal basis vectors. Thus any vector (or function) can be expanded in terms of a set of orthogonal basis vectors (or functions).**

**Normalization of functions:**

**The scalar product** $\vec{A} \bullet \vec{A} = A^2$ **gives the square of the length of a vector), where the length of a vector is also called the norm of a vector.**

**What is the norm of a unit vector?**

**How can we find the norm of a function A (*x*) on the interval (a, b)?**

**Answer:** $\int_a^b A^*(x) A(x) dx = \int_a^b |A(x)|^2 dx = N^2 \Rightarrow$ ***N* is called the norm of the function over the interval (a, b).**

**Can we get a normalized function?**

**Answer: Yes, we just take** $\frac{A(x)}{N}$ **as a normalized function where** $N^{-1}$ **is called the normalization factor.**

**Exercise:** **a) What is the norm of the function sin *nx* on (0, π)?**

**Answer:** **Firstly, we find** $\int_0^{\pi} |\sin nx|^2 dx = \frac{\pi}{2}$ **and then the norm is** $\sqrt{\frac{\pi}{2}}$ **.**

**b) Find the normalized function.**

**Answer:** **The normalized function is** $\sqrt{\frac{2}{\pi}} \sin nx$**. [How can you be so sure of that?!**

## Orthonormal set of vectors:

**The unit vectors** $\hat{i}, \hat{j}$ **and** $\hat{k}$ **are a set of orthonormal vectors because they are orthogonal to each other and each has a norm equal 1.**

**Note:** **Any set of normalized and orthogonal functions is called orthonormal (e.g.** $\sqrt{\frac{2}{\pi}} \sin nx$ **is an orthonormal set of basis functions on (0 ,π).**

**Any function (or a vector in vector space) like f (x) can be expanded on (0, π) in a Fourier sine series as**

$$f(x) = \sum_n B_n \sqrt{\frac{2}{\pi}} \sin nx$$

**Here *f*(*x*) is considered a vector (or a function) with components** $B_n$ **while** $\sqrt{\frac{2}{\pi}} \sin nx$ **are the basis vectors.**

**[Note: In Q. M. a physical system is expressed as either a state function or a state vector].**

## Normalization of the Legendre polynomials:

**What is the norm of** $P_\ell(x)$ **on the interval (-1, 1)?**

**Answer:** **The norm of** $P_\ell(x)$ **is.** $\int_{-1}^{1}[P_\ell(x)]^2\,dx = \frac{2}{2\ell+1}$

**Proof:** **Use the identity** $xP'_\ell(x) - P'_{\ell-1}(x) = \ell P_\ell(x)$ ***and multiply both sides by*** $P_\ell(x)$ **and integrate to get**

$$\ell\int_{-1}^{1}[P_\ell(x)]^2\,dx = \int_{-1}^{1} xP_\ell(x)P'_\ell(x)dx - \int_{-1}^{1} P_\ell(x)P'_{\ell-1}(x)dx$$

**The last integral is zero (See problem 4 of section 7). Use the method of integration by part to find the 1st integral on R. H. S of the equation as:**

$$\int_{-1}^{1} xP_\ell(x)P'_\ell(x)dx = 1 - \frac{1}{2}\int_{-1}^{1}[P_\ell(x)]^2\,dx$$

**[Hint: to reach this answer you may need the identity )** $P_\ell(-1) = (-1)^\ell$ **](See problem 2, section 2).**
**Substitute this result back into the previous equation to get:**

$$\therefore \int_{-1}^{1}[P_\ell(x)]^2\,dx = \frac{2}{2\ell+1}$$

**Q. E. D.**

## Orthonormal set of Legendre functions:

a. **The functions** $\sqrt{\frac{2\ell+1}{2}}P_\ell(x)$ **form an orthonormal set of functions n (-1, 1).**

b. $\int_{-1}^{1}\sqrt{\frac{2\ell+1}{2}}P_\ell(x)\sqrt{\frac{2m+1}{2}}P_m(x)dx = \delta_{\ell m}\begin{cases}0 & if \quad \ell \neq m \\ 1 & if \quad \ell = m\end{cases}$**, where** $\delta_{\ell m}$ **is called Kronecker delta.**

**Reminder for suggested problems: Solve problems (6.2, 6.5, 6.6), 7.4, (8.1, 8.2, 8.5)**