Other ways of obtaining Legendre Polynomials:

1) Rodrigues Formula: $$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

***Exercise:* Find $P_o(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, and $P_4(x)$ from Rodrigues formula and compare your results with those obtained previously. (See problem 3, section 4).**

2) Generating Function: $\Phi(x,h) = (1-2xh+h^2)^{-1/2}$ $\qquad |h| < 1$

and $$\Phi(x,h) = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

***Proof:* Put $2xh - h^2 = y$ and expand $(1-y)^{-1/2}$ in power of y to get:**

$\Phi(x,h) = 1 + \frac{1}{2}y + \frac{\frac{1}{2}\bullet\frac{3}{2}}{2!}y^2 + \bullet\bullet\bullet$ **and substitute back $y = 2xh - h^2$. After simple rearrangements of terms you may have:**

$$\Phi(x,h) = 1 + xh + \frac{h^2}{2}(3x^2-1) + \bullet\bullet\bullet$$

Recalling the obtained expressions for $P_o(x)$, $P_1(x)$, $P_2(x)$,…..etc., the generating function can be rewritten as:

$$\Phi(x,h) = P_\circ(x) + hP_1(x) + h^2P_2(x) + \bullet\bullet\bullet = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x)$$

***Question:* Is this a full proof that $P_\ell(x)$ are really Legendre Polynomials?**

***Answer:* No it is not, but this is a strict verification of the 1st three terms. However, to prove that $P_\ell(x)$ are Legendre Polynomials:**

i) $P_\ell(x)$ must satisfy the Legendre DE.(This will be left to be proved by the student).

ii) $P_\ell(x)$ should have the property $P_\ell(1) = 1$.
(See problem 2, section 4).

[*Hint:* To solve problem 2, section 4, put x = 1 in the equations $\Phi(x,h) = (1-2xh+h^2)^{-1/2}$ & $\Phi(x,h) = P_\circ(x) + hP_1(x) + h^2P_2(x) + \bullet\bullet\bullet$, then equate them after simple arrangements].

Recursion Relations for Legendre polynomials*:*

a) $\ell P_\ell(x) = (2\ell - 1)xP_{\ell-1}(x) - (\ell - 1)P_{\ell-2}(x)$

b) $xP'_\ell(x) - P'_{\ell-1}(x) = \ell P_\ell(x)$

c) $P'_\ell(x) - xP'_{\ell-1}(x) = \ell P_{\ell-1}(x)$

d) $(1-x^2)P'_\ell(x) = \ell P_{\ell-1}(x) - \ell x P_\ell(x)$

e) $(2\ell + 1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x)$

Properties of Legendre polynomials*:*

1. ***The general behavior of Legendre polynomials can be shown by sketching graphs of $P_o(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ from x = - 1 to x = 1. (See problem 2, section 2).***
2. $P_\ell(-1) = (-1)^\ell$ ***(See problem 2, section 2).***

   ***Exercise:* Find** $P_\ell(0)$
3. $\int_{-1}^{1} x^m P_\ell(x)dx = 0$ ***if*** $m\langle \ell$ ***(see problem 4, section 4).***

[Some other properties will be shown later on].

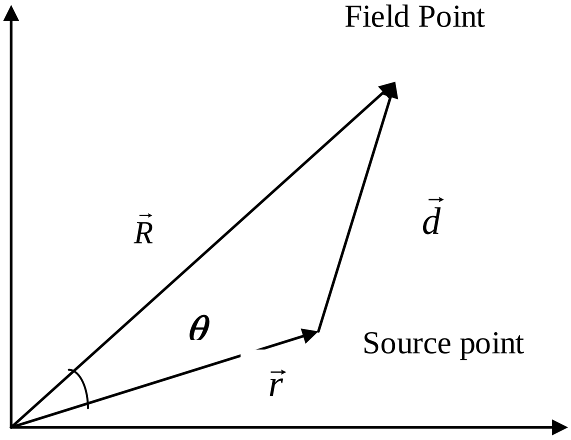
Suggested problems:* *Chapter 12, section 5 (4, 5, 6, 9, 11).

Expansion of a potential:
(An application for Legendre polynomials)

Φ(x, h) is useful in problems dealing with the potential of the type $V \sim \frac{1}{d}$***, where d is the distance between the source and field points. (e.g. gravitational or electrostatic potential). This potential can be written as*** $V = \frac{K}{d}$***, where K is an appropriate constant that depends on the type of potential. The distance d, shown in the diagram, may be expressed by:***

$|\vec{d}| = |\vec{R} - \vec{r}|$

Using the cosine law we get:



$$d = (R^2 + r^2 - 2Rr\cos\theta)^{1/2}$$
$$= R(1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2})^{1/2}$$

Put $h = \frac{r}{R}$ ***and x = cos*** θ

$$\therefore V = \frac{K}{R}(1 - 2hx + h^2)^{-1/2}$$

For the electrostatic problem with a single charge: K= kq

$$\therefore V = \frac{K}{R}\Phi(x, h)$$

But $\Phi(x,h) = \sum_{\ell=0}^{\infty} h^{\ell} P_{\ell}(x)$

$$\therefore V = \frac{K}{R}\sum_{\ell=0}^{\infty} h^{\ell} P_{\ell}(x)$$

Or $\boxed{V = K\sum_{\ell=0}^{\infty} \frac{r^{\ell}}{R^{\ell+1}} P_{\ell}(\cos\theta)}$

#For several charges q_i: (Discrete charge distribution): K = kq$_i$

$$\boxed{V = k\sum_{\ell=0}^{\infty}\sum_{i} q_i \frac{r_i^{\ell}}{R^{\ell+1}} P_{\ell}(\cos\theta)}$$

 # For a continous charge distribution: $\sum_i q_i \Rightarrow \int dq \to \int \rho d\tau$***,***

and the potential can be expressed by:

$\boxed{V = k\sum_{\ell} \frac{1}{R^{\ell+1}} \iiint r^{\ell} P_{\ell}(\cos\theta)\rho d\tau}$ ***, where ρ is the volume charge density.***

***ecial cases*:**

Monopole case (single charge), put ℓ =0 you may get

$V = \frac{K}{R}Q$ ***, where*** $Q = \int \rho d\tau$ ***is the total charge.***

Dipole case, put ℓ=1 you get $V = \frac{k}{R^2}\iiint r\cos\theta\rho d\tau$

lote:* Other cases of ℓ=2 (Quadrupole) and ℓ=3 (Octopole)
ill not be tackled here].