The Legendre's Equation

The Legendre's DE is written as:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0,$$

where ℓ is a real constant, $p(x) = -\frac{2x}{1-x^2}$ and $q(x) = \frac{\ell(\ell+1)}{1-x^2}$.
The point $x = \pm 1$ represents a singularity (a regular singular point), since $(1-x^2)\,p(x)$ is finite and $(1-x^2)\,q(x)$ is also finite. However, it must be noted that $x = 0$ is ordinary point.

Now Frobenius method can be applied:

Put $\ell(\ell+1) = \lambda$ and substitute the assumed solution

$y = \sum_{n=0}^{\infty} a_n x^{m+n}$ into the DE to obtain:

$$\sum_{n=0}^{\infty} a_n (m+n)(m+n-1)x^{m+n-2} - \sum_{n=0}^{\infty} a_n (m+n)(m+n-1)x^{m+n}$$

$$-2\sum_{n=0}^{\infty} a_n (m+n)x^{m+n} + \lambda\sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

which gives

$$a_0 m(m-1)x^{m-2} + a_1 m(m+1)x^{m-1} + \sum_{n=2}^{\infty} a_n (m+n)(m+n-1)x^{m+n-2}$$
$$-\sum_{n=0}^{\infty}[a_n (m+n)(m+n-1) + 2a_n (m+n) - \lambda a_n]x^{m+n} = 0,$$

Writing $n+2$ for n in the 1st summation to get:

$$a_0 m(m-1)x^{m-2} + a_1 m(m+1)x^{m-1} + \sum_{n=0}^{\infty}[a_{n+2}(m+n+2)(m+n+1)$$
$$-a_n (m+n)(m+n-1) - 2a_n (m+n) - \lambda a_n]x^{m+n} = 0.$$

Equating the coefficient of all powers of x to zero to obtain:

$a_\circ m(m-1)=0$, [This is called the indicial equation that determines the values of the index *m*. This equation gives the roots *m = 0* and *m = 1]*.

$$a_1 m(m+1)=0\,,$$

$$a_{n+2}=\frac{(m+n)(m+n-1)+2(m+n)-\lambda}{(m+n+2)(m+n+1)}a_n$$

Case (1): *m = 0*

a_o *and* a_1 are arbitrary constants.

$$\therefore a_{n+2}=\frac{n(n-1)+2n-\lambda}{(n+2)(n+1)}a_n$$

put $\ell(\ell+1)=\lambda$ and take the values of n in order to get:

$$a_2=-\frac{\ell(\ell+1)}{2}a_\circ\,,\qquad a_3=\frac{2-\ell(\ell+1)}{3\bullet 2}a_1=-\frac{(\ell+2)(\ell-1)}{3!}a_1\,,$$

$$a_4=\frac{2+4-\ell(\ell+1)}{4\bullet 3}a_2=\frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}a_\circ\,,$$

$$a_5=\frac{6+6-\ell(\ell+1)}{5\bullet 4}a_3=\frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}a_1$$

$\therefore$ The general solution is:

$$y=a_\circ[1-\frac{\ell(\ell+1)}{2!}x^2+\frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4-\bullet\bullet\bullet]$$
$$+a_1[x-\frac{(\ell-1)(\ell+2)}{3!}x^3+\frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}x^5-\bullet\bullet\bullet]$$

These two independent series solutions are called Legendre functions.

Case (2): *m = 1*

Again as mentioned before, this gives a dependent solution.

Hence the general solution is same as mentioned above.

Now, for integral ℓ, we are seeking a solution which converges at $x = \pm 1$. In addition we need a solution for $|x| < 1$.

The obtained general solution, at x = ±1, will become a set of Legendre polynomials when a set of values of integral ℓ is given, as follows:

a) *For ℓ =0 with x =1 we can show that the odd series diverges (by using the ratio test) like* $(1+\frac{1}{3}+\frac{1}{5}+\bullet\bullet\bullet\bullet)$.

[Note: Try the ratio test $\frac{a_{n+2}}{a_n}=\frac{n(n-1)+2n-\ell(\ell+1)}{(n+2)(n+1)}$ *and start with n=1, and then other odd n's.].*

b) For the same ℓ but with even n, use the ratio test e.g. for n=2 we get the ratio a_4/a_2 not defined. All other even n, give non-defined ratios as well. Only a_o survives such that $y = a_o$ at x = 1. Here, when y = 1 ⇒ a_o = 1. Thus the solution y =1 can be named as a polynomial $\boxed{P_\circ(x)=1}$. *This is Legendre polynomial* $P_\ell(x)$ *for* $\ell = 0$.

For ℓ =1 with x = 1, the a_o series (even series) can be shown to be divergent. But a_1 series (odd series) will give $y = a_1 x$. Again when y=1 and for x = 1 ⇒a_1=1.

Thus $\boxed{P_1(x)=x}$.

For ℓ =2 with x = 1, the odd series diverges while the even series becomes $y=a_\circ[1-\frac{2\bullet 3}{2!}x^2+\bullet\bullet\bullet]$. *Hence at x = 1 and y = 1 ⇒ a_o= -1/2.*

$$\boxed{\therefore P_2(x)=\frac{1}{2}(3x^2-1)}.$$

For ℓ *=3, only the odd series survives and we get* $y = a_1(x - \frac{5}{3}x^3)$,

for x = 1 and y = 1 ⇒ a_1*= - 3/2.*⇒ $\boxed{\therefore P_3(x) = \frac{3}{2}(\frac{5}{3}x^3 - x)}$.

Note: *Other Legendre Polynomials can be left as exercise for the students. Try to find*

$$\boxed{P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)}$$

and

$$\boxed{P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)}.$$

Remarks:

a. *The second solution for each* ℓ *which is infinite series at x=* ± 1*, is convergent for* $|x| < 1$*. This latter solution is called a Legendre function of second kind* $Q_\ell(x)$*. The functions* $Q_\ell(x)$ *are not used as frequently as* $P_\ell(x)$.

b. *For fraction (non-integral)* ℓ *both solutions are infinite series and again these occur less frequently in applications.*

c. *By solving the Legendre DE, we actually have solved what is called the eigenvalue problem. That is, the values of* ℓ*, namely, 0, 1, 2, and 3 are called eigenvalues and the corresponding solutions* $P_\ell(x)$ *are called eigenfunctions.*