Series Solution of ODE

The Frobenius Method:

This method is adopted, if the following type of DE needs a solution; $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0$, where *p(x)* and *q(x)* are given functions of *x*.

[Note: There are some exceptions in which the first term of this DE does not exist].

We are seeking solutions to this ODE in the neighborhood of *x* = 0.

1. When *x* = 0, this may be called *Ordinary Point* of the DE $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=0$, because p(0) and q(0) are finite when *x* = 0.
2. When *x* = 0, this may be called a *Regular Singular Point* of the DE $\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\frac{y}{x^2}=0$, because *xp(x)* and $x^2q(x)$ remain finite at *x* = 0. [*i.e.* $p(x)=3/x$, $q(x)=1/x^2$ such that $xp(x)=3$ and $x^2q(x)=1$, they are both finite when *x = 0*].
3. When *x* = 0, this may be called an *Irregular Singular Point* of the DE $\frac{d^2y}{dx^2}+\frac{1}{x^2}\frac{dy}{dx}+xy=0$, because either of the conditions in (1) and (2) is not satisfied. Frobenius method of solution about *x = 0* can not be applied.

The general solution: Using the Frobenius method is to assume a series solution of the type:

$$y=\sum_{n=0}^{\infty}a_nx^{m+n}=x^m(a_\circ+a_1x+a_2x^2+\bullet\bullet\bullet\bullet+a_nx^n),$$

where a_o, a_1, a_2, …… a_n and m are constants to be determined. This series is called the general power series.

[Note: a_o is taken as arbitrary constant (*i.e.* $a_o \neq 0$)]

The possible values of *m*: (It may take a positive or negative number and it may be a fraction)

When *m* takes a negative number (like, for instance, *m*= *-2*), the series solution may look like:

$$y = x^{-2}\cos x = x^{-2}(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \bullet\bullet\bullet\bullet)$$

When *m* takes a fraction (like, for instance, *m*= ½), the series solution may be like $y = x^{1/2}\sin x = x^{1/2}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \bullet\bullet\bullet\bullet)$

Example: Solve the DE $\frac{d^2y}{dx^2} - xy = 0$ about *x* = *0*.

Solution: Here we have *q (x)*= - *x* and we are seeking a solution about an ordinary point (*x* = *0*).

Now assume the general solution $y = \sum_{n=0}^{\infty} a_n x^{m+n}$, and substitute it into the DE to obtain:

$$\sum_{n=0}^{\infty} a_n (m+n)(m+n-1)x^{m+n-2} - \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0,$$

which gives:

$$a_0 m(m-1)x^{m-2} + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m$$
$$+\sum_{n=3}^{\infty} a_n (m+n)(m+n-1)x^{m+n-2} - \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

Writing n+3 for n in the 1st summation term to get:

$$a_0 m(m-1)x^{m-2} + a_1 m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m$$
$$+[\sum_{n=0}^{\infty} a_{n+3}(m+n+3)(m+n+2) - \sum_{n=0}^{\infty} a_n]x^{m+n+1} = 0$$

Equating the coefficients of all powers of *x* to zero, we have:

$a_\circ m(m-1) = 0$, [this is called the indicial equation that determines the values of the index *m*. This equation gives the roots *m = 0* and *m = 1]*.

$$a_1 m(m+1)=0,$$
$$a_2(m+2)(m+1)=0,$$

and

$$a_{n+3}=\frac{a_n}{(m+n+3)(m+n+2)}$$, (where *n=0, 1, 2,* ……ect).

[This is called the recursion relation].

Now, we have $\boldsymbol{a_o \neq 0}$, *m = 0* or *m = 1.*

Case (1): *m = 0*

$a_1 m(m+1)=0$, here $\boldsymbol{a_1}$ ***must not equal*** **zero.**

But $a_2(m+2)(m+1)=0$ shows that $\boldsymbol{a_2=0}$.

∴ The recursion relation gives:

$$a_3=\frac{a_\circ}{3\bullet 2},\qquad a_4=\frac{a_1}{4\bullet 3},\qquad a_5=\frac{a_2}{5\bullet 4}=0$$

$$a_6=\frac{a_\circ}{6\bullet 5\bullet 3\bullet 2},\qquad a_7=\frac{a_1}{7\bullet 6\bullet 4\bullet 3},\qquad a_8=\frac{a_2}{8\bullet 7\bullet 5\bullet 4}=0,$$

and so on.

∴ The solution is:

$$y=a_\circ\left(1+\frac{x^3}{3\bullet 2}+\frac{x^6}{6\bullet 5\bullet 3\bullet 2}+\bullet\bullet\right)+a_1\left(x+\frac{x^4}{4\bullet 3}+\frac{x^7}{7\bullet 6\bullet 4\bullet 3}+\bullet\bullet\right),$$

where $\boldsymbol{a_o}$ and $\boldsymbol{a_1}$ ***are arbitrary independent constants.***

Case (2)***:*** *m = 1*

$\boldsymbol{a_o \neq 0}$ as assumed before.

$a_1 m(m+1)=0$ gives $\boldsymbol{a_1=0}$.

Also

$a_2(m+2)(m+1)=0$ gives $\boldsymbol{a_2=0}$.

$a_3 = \frac{a_\circ}{4 \bullet 3}$, $a_4 = \frac{a_1}{4 \bullet 3} = 0$, $a_5 = \frac{a_2}{5 \bullet 4} = 0$

$a_6 = \frac{a_o}{7 \bullet 6 \bullet 4 \bullet 3}$, $a_7 = \frac{a_1}{8 \bullet 7 \bullet 5 \bullet 4} = 0$, $a_8 == 0$,

and so on.

Thus the solution corresponding to *m = 1* is :

$$y = a_\circ x(1 + \frac{x^3}{4 \bullet 3} + \frac{x^6}{7 \bullet 6 \bullet 4 \bullet 3} + \bullet\bullet)$$

[Apart from an arbitrary constant, this is just the same as the first series in the previous solution of case (1)].

Therefore, the general solution is:

$$y = a_\circ (1 + \frac{x^3}{3 \bullet 2} + \frac{x^6}{6 \bullet 5 \bullet 3 \bullet 2} + \bullet\bullet) + a_1 (x + \frac{x^4}{4 \bullet 3} + \frac{x^7}{7 \bullet 6 \bullet 4 \bullet 3} + \bullet\bullet)$$

[Note: Frobenius method does not always give two independent series solutions].

Conclusions:

a) If the root of the indicial equationdiffer by an integer (with certain exceptions), there is only one independent series solution.

b) If the two roots of the indicial equation are the same, not more than one independent series solution exists.

Exercise*:* Solve the DE $4x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0$ **after specifying the type of the point.**