Complete elliptic integrals

When $\phi = \frac{\pi}{2}$, the elliptic integrals are called the complete elliptic integrals of first and second kinds.

$$K(k) = F(k, \frac{\pi}{2}) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}$$

$$E(k) = E(k, \frac{\pi}{2}) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \phi}\, d\phi$$

[Note: These integrals have special tables which are more accurate than the tables of F(*k, φ*) and E(*k, φ*)].

Other properties of Legendre forms of elliptic integrals:

Question: How can we find integrals with various values of φ?

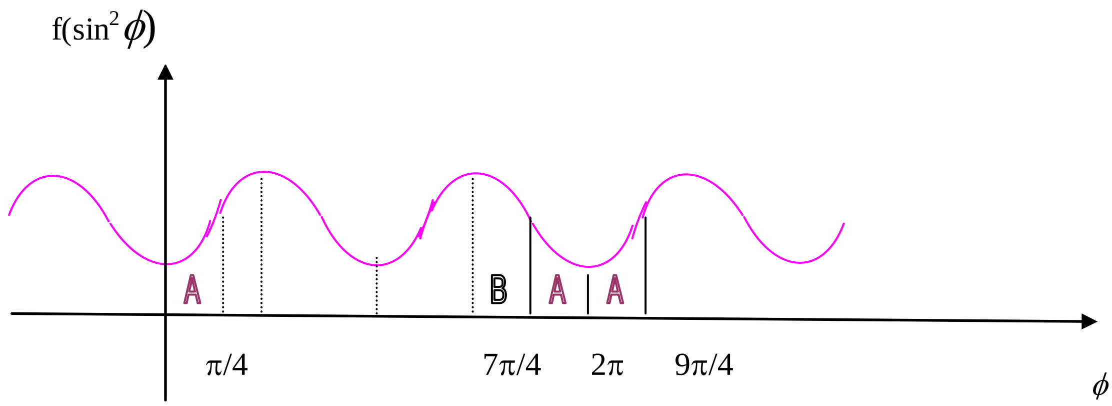
Answer: We must always integrate over a number of π (not π/2) intervals and then add or subtract the correct integral over an interval of length less than π/2.

1. **Using the definition of complete elliptic integrals, we can show that**

$$F(k, n\pi \pm \phi) = 2nK \pm F(k, \phi)$$

$$E(k, n\pi \pm \phi) = 2nE \pm E(k, \phi)$$

Proof: **This can be proved by having a close insight into the shown diagram:**



The elliptic integrals are both functions of $\sin^2\phi$, namely $f(\sin^2\phi)$. If we plot this function such that ϕ between 0 and $\pi/2$ and that of ϕ between $\pi/2$ and π will give the same value. Thus for ϕ between 0 and π is one period of $f(\sin^2\phi)$. The rest of the graph repeats itself.

Remember that the area under the curve $\int f(\sin^2\phi)d\phi$ could be either F*(k, φ)* or E((*k, φ).* Now we can find the area under the curve for ϕ between 0 and $9\pi/4$ as follows:

$$\int_0^{9\pi/4} = \int_0^{2\pi} + areaA = \int_0^{2\pi} + \int_0^{\pi/4} = 4\int_0^{\pi/2} + \int_0^{\pi/4}$$

Also the area under the curve for ϕ between 0 and $7\pi/4$ as follows:

$$\int_0^{7\pi/4} = \int_0^{2\pi} - areaA = \int_0^{2\pi} - \int_0^{\pi/4} = 4\int_0^{\pi/2} - \int_0^{\pi/4}$$

[Warning: **You have to be very careful here not to confuse this later area with** $\int_0^{7\pi/4} \neq \int_0^{3\pi/2} + \int_0^{\pi/4}$ **].**

2. **The elliptic integrals can be shown to be odd functions of ϕ, namely:**

$$F(k,-\phi) = -F(k,\phi)$$
$$E(k,-\phi) = -E(k,\phi)$$

3. **When the lower limit in the elliptic integrals is different from zero, they can be expressed as follows:**

$$\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \int_0^{\phi_2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} - \int_0^{\phi_1} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = F(k,\phi_2) - F(k,\phi_1)$$

Similarly

$$\int_{\phi_1}^{\phi_2} \sqrt{1-k^2\sin^2\phi}\, d\phi = E(k,\phi_2) - E(k,\phi_1)$$

Exercise 1: Evaluate $I = \int_0^{\infty} \sqrt{\frac{9+8x^2}{(1+x^2)^3}}\, dx$,

[Hint: take E(1/3)=1.525].

Solution: **Put $x = \tan\phi$ and proceed.**

Exercise 2: Evaluate $I = \int_0^{\pi/6} \frac{d\alpha}{\sqrt{1-4\sin^2\alpha}}$,

[Hint: take K(1/2)=1.688].

Solution: **Put $4\sin^2\alpha = \sin^2\phi$ and proceed…**

Example: The problem of simple pendulum for large angles.

Solution:

Recalling the DE of motion $(\theta^{\bullet})^2 = \frac{2g}{\ell}\cos\theta + C$.

By taking a swing of any amplitude, say α**, rather than** $\pi/2$ **such that** $\theta^{\bullet} = 0$ **at** $\theta = \alpha$**, we get** $C = -\frac{2g}{\ell}\cos\alpha$.

$$\therefore (\theta^{\bullet})^2 = \frac{2g}{\ell}(\cos\theta - \cos\alpha).$$

It must be noted that the period of a swing from $-\alpha$ ***to*** α **and back is** T_α**. Thus the limits can be summarized as :**

$$\begin{cases} \theta = 0 \Rightarrow \theta = \alpha \\ t = 0 \Rightarrow t = \frac{T_\alpha}{4} \end{cases}$$

The above DE can be rewritten as:

$$\int_0^{\alpha} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \int_0^{T_\alpha/4} \sqrt{\frac{2g}{\ell}}\,dt = \sqrt{\frac{2g}{\ell}}\,\frac{T_\alpha}{4}$$

To obtain a final result for T_α **we should solve problem 17, section 12, chap.11.**

We need to evaluate the integral $I = \int_0^{\alpha} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$.

Put $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi$ **and change the limit of the integral such that for** $\theta = 0 \Rightarrow \phi = 0$ **and for** $\theta = \alpha \Rightarrow \phi = \pi/2$**. We have**

$$I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \left(\frac{2\sin\frac{\alpha}{2}\cos\phi}{\cos\frac{\theta}{2}} \right) \frac{d\phi}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}},$$

But $\cos\frac{\theta}{2} = \sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}$

and $\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\sin^2\phi} = \sin\frac{\alpha}{2}\cos\phi$.

$$I = \sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}},$$

$$I = \sqrt{2} F(\sin\frac{\alpha}{2}, \frac{\pi}{2}) = \sqrt{2} K(\sin\frac{\alpha}{2})$$

$$\boxed{\therefore T_\alpha = 4\sqrt{\frac{\ell}{g}} K(\sin\frac{\alpha}{2})}$$

Special cases:

i) **For α not too large (i.e. $\alpha < \pi/2 \Rightarrow \sin^2 \alpha/2 < ½$)**

[You have to solve problem 1, section 12, chap. 11].

You might get a good approximation for T_α when you use the binomial expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \bullet\bullet\bullet\bullet$$

Thus $F(k,\frac{\pi}{2}) = \int_0^{\pi/2} (1-\sin^2\frac{\alpha}{2}\sin^2\phi)^{-1/2} d\phi$ **can be written as:**

$$F(k,\frac{\pi}{2}) \approx \int_0^{\pi/2} (1 + \frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi + \frac{3}{8}\sin^4\frac{\alpha}{2}\sin^4\phi + \bullet\bullet\bullet)d\phi$$

$$\therefore T_\alpha = 4\sqrt{\frac{\ell}{g}}K(\sin\frac{\alpha}{2})\{\frac{\pi}{2}(1 + \frac{1}{2}\bullet\frac{1}{2}\sin^2\frac{\alpha}{2} + \frac{3}{8}\bullet\frac{3}{8}\sin^4\frac{\alpha}{2} + \bullet\bullet)\}$$

ii*) For small enough** α ⇒ ***sin $\alpha/2 \approx \alpha/2$**, hence**

$$T_\alpha = 2\pi\sqrt{\frac{\ell}{g}}(1 + \frac{\alpha^2}{16} + \bullet\bullet\bullet)$$

iii) **For very small** α **we will reach to the approximate result, that is,**

$$T_\alpha = 2\pi\sqrt{\frac{\ell}{g}}$$

Exercise: Evaluate the integral $I = \int_0^x \sqrt{\frac{10-5x^2}{1-x^2}}dx$.

[Hint: you may reach a step with $I = \sqrt{10}E(k,\phi)$ **, where** $k = \frac{1}{\sqrt{2}}$**].**