Elliptic Integrals

Integrals can be solved in two ways

Analytically
(In terms of elementary and complicated functions)
{Standard Integrals}

Numerically
(By approximation methods)
{Non-standard integrals}

e.g. $\int \frac{dx}{P\sqrt{Q}}$ **(*P* may be a linear function of *x* while *Q* is polynomial)**

Integrals can be expressed explicitly in terms of elementary functions(sin*x*, cos*x*, log*x*,…).
When *Q* is a polynomial of degree 2 or less the integral can be defined as:
e.g.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$$

for |*x*/< *a*

Integrals can be expressed in terms of more complicated functions like elliptic function and Γ function.
When *Q* is a polynomial of *x* of degree 3 or 4,this integral is said to be of elliptic type, e.g.

$$\int_o^1 \frac{dx}{\sqrt{(1-x^2)(2-x^2)}}$$

Forms of Elliptic Integrals

Legendre forms

1. First kind of elliptic integrals:

$$F(k,\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

2. Second kind of elliptic integrals:

$$E(k,\phi)=\int_0^{\phi}\sqrt{1-k^2\sin^2\phi}\,d\phi$$

Jacobi forms

First kind of elliptic integrals:

$$F(k,\phi)=\int_0^{x}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

2. Second kind of elliptic integrals:

$$E(k,\phi)=\int_0^{x}\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

Where *k* is called the modulus and ϕ is called the amplitude.

k= sin θ where $0\le\theta\le\frac{\pi}{2}$ and $0\le\phi\le 2\pi$.

Also *x*= sin ϕ in Jacobi forms.

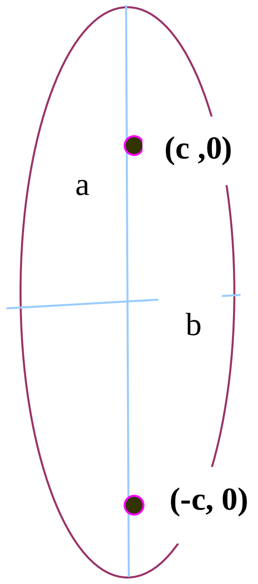
Basic definitions and properties of Legendre elliptic integrals:

Definition:

Length of the ellipse $=4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2\phi}\,d\phi,$

Where *x*= *a* cos ϕ and y= *b* sin ϕ and $0\le\phi\le 2\pi$.

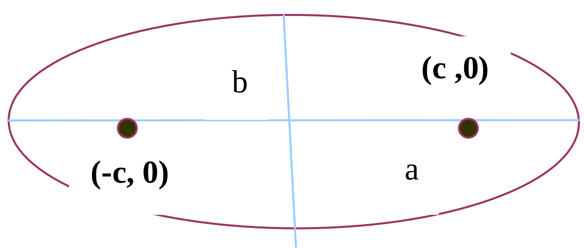
Here *e* is called the eccentricity of ellipse, which can be renamed as *k*. Thus $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$**, where *a* and *b* are the semimajor and semiminor axes, respectively, while c is the focus distance on y-axis, as shown in the figure. (Here *a* > *b*).**



The equation of ellipse, here, is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

Example: Find the arc length of an ellipse. This is the problem that gave elliptic integrals their name.
[Hint: take the case a > b where *x= a sin* ϕ and *y= b* cos ϕ].

The equation of ellipse, here, is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Solution:

The length of an arc of ellipse is $\int_b^a ds$ **from b** $\Rightarrow$ ***a*.**

But $ds = \sqrt{(dx)^2 + (dy)^2}$

$$ds = \sqrt{(a\cos\phi)^2 + (b\sin\phi)^2}\,d\phi$$

For *x*= *0* and *y*= *b* ⇒ ϕ= 0

For *x*= *a* and *y*= *0* ⇒ ϕ= π/2.

Thus the limit from b⇒ *a* is ϕ= 0 ⇒ π/2.

$$\int_b^a ds = \int_0^{\pi/2} \sqrt{a^2(1-\sin^2\phi) + b^2\sin^2\phi}\,d\phi$$

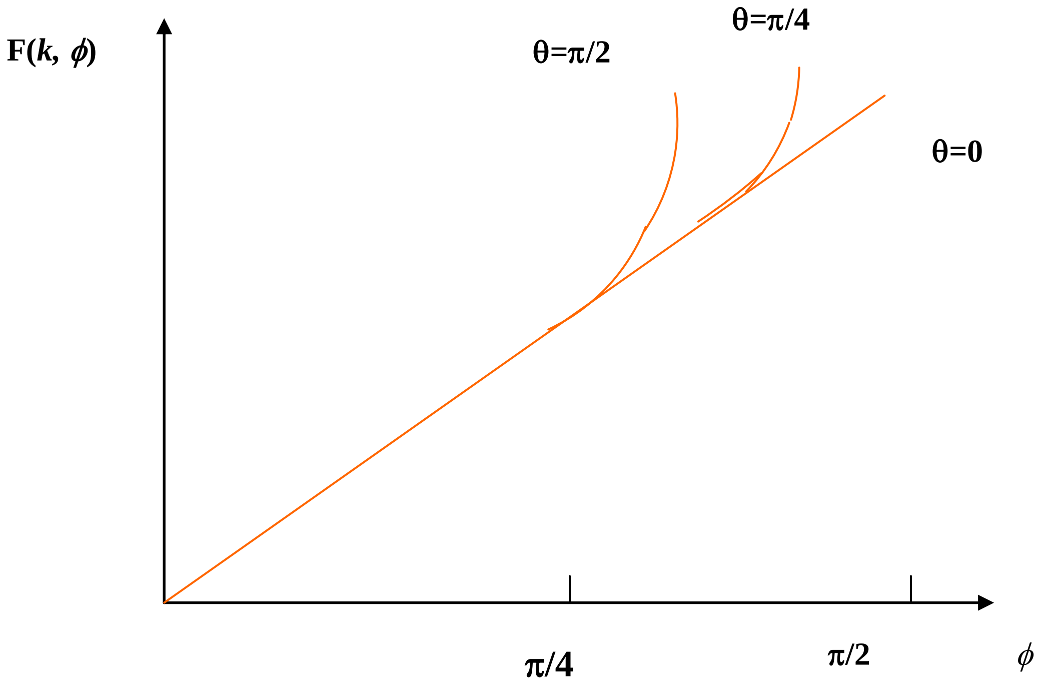
$$\int_b^a ds = a\int_0^{\pi/2} \sqrt{1 - \frac{a^2 - b^2}{a^2}\sin^2\phi}\,d\phi$$

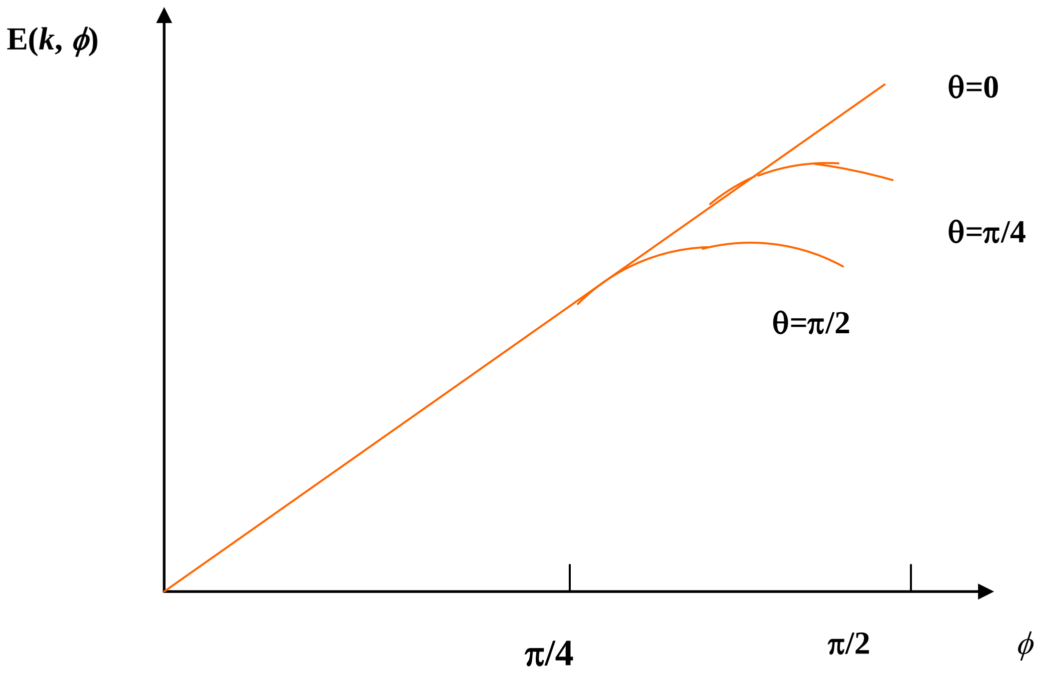
As mentioned above $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$ **for a > b. Also *e* can be renamed as *k*.**

$$\int_b^a ds = a\int_0^{\pi/2} \sqrt{1 - k^2\sin^2\phi}\,d\phi$$

Thus length of arc of ellipse $= aE(k, \frac{\pi}{2})$**.**

Properties of elliptic integrals:(Legendre forms)





Note: The values of the functions F(k, ϕ) and E(k, ϕ) are tabulated for large numbers of k and ϕ.