Integral Transforms

The Laplace Transform

The Laplace transform of f(t) can be defined as

$$F(p) = \int_0^\infty f(t)e^{-pt}dt$$

When we want to solve problems by using the Laplace transform, it is very convenient to have a table of corresponding *F(t)* and *F(P)*.

Example: For F(t) = 1 with P>0

$$F(P) = \int_o^\infty e^{-pt}dt = -\frac{1}{p}e^{-pt}{}_o\int^\infty$$

$$= \frac{1}{p}$$

Note: If ***p*** is complex, then the real part of p (Rep) must be positive.

Example: *For F*(t) = e^{-at}

$$F(p) = \int_0^\infty e^{-(a+p)t}dt + \frac{1}{p+a}$$

Re *(p+a)*>0

Example: The Laplace transform of a sum of two functions is the sum of their Laplace transforms

$$L[f(t)+g(t)] = \int_0^\infty [F(t)+g(L)]e^{-pt}dt$$

$$= \int_0^\infty f(t)e^{-pt}dt + \int_0^\infty g(t)e^{-Pt}dt$$

= *L(f)* + *L(g)*

Example: *For f(t) = sin at* $\mathrm{Re}(p) > |\mathrm{Im}\, a|$

Subtract $\int_0^\infty e^{(a-p)t}dt - \int_0^\infty e^{-((a+p)t}$

$$= \frac{2ia}{p^2 + a^2}$$

$$\therefore L\left(\frac{e^{iat} - e^{-iat}}{2i}\right) = \frac{a}{p^2 + a^2}$$

Example: For $\mathrm{f(t)} = \cos(\mathrm{at}) = \dfrac{e^{iat} + e^{-iat}}{2}$

$$\int_0^\infty e^{(ia-p)}dt + \int_0^\infty e^{-(ia+p)t}dt = 2\frac{p}{p^2 + p^2}$$

$$\therefore L\left(\frac{e^{iat} + e^{-iat}}{2}\right) = \frac{p}{p^2 + a^2}$$

Example: Find the Laplace transform of *t* sin (*at*) with

$$\mathrm{Re}(p) > |\mathrm{Im}\, a|$$

$$\mathrm{L}(\cos \mathrm{at}) = \frac{p}{p^2 + a^2}$$

$$\int_0^\infty e^{-pt} \cos atdt = \frac{p}{p^2 + a^2}$$

Differentiate (both sides) with refrence to ***a***

$$\int_0^\infty -te^{-pt} \sin atdt = \frac{-2ap}{(p^2 + a^2)^2}$$

$$\Rightarrow \int_0^\infty e^{-pt} t \sin atdt = \frac{2p^a}{(p^2 + a^2)^2}$$

$$L[CF(t)] = \int_0^\infty Cf(t)e^{-pt}$$

$$= c\int_0^\infty F(t)e^{-pt} dt = CL(f)$$

The Laplace transform is a linear operator

Example: For f(t)= e^{iat} (replacing *a* by *ia)*

= *cos* (*at*) + i *sin*(*at*)

$$F(p) = \int_0^\infty e^{iat} e^{-pt} dt$$

$$= \int_0^\infty e^{(ia-p)t} dt = \frac{1}{ia-p} e^{(ia-p)t} \int_0^\infty$$

$$= \frac{1}{ia-p}[-1]$$

$$= \frac{1}{p-ia} = \frac{p+ia}{p^2+a^2}$$

$$\frac{p}{p^2+a^2} + \frac{ia}{p^2+a^2}$$

R e(p-ia)>0

Another method

L (*cos (at*) + *i sin(at*)) = L (*cos at*) + i L (*sin at*)

$$= \frac{p}{p^2+a^2} + i\frac{a}{p^2+a^2}$$

Example for f(t) = e^{-iat} (replacing *a* by –*ia*)

$$F(p) = \int_0^\infty e^{-(ia+p)t} dt = \frac{1}{-(ia+p)} e^{-(ia=P)T} \int_0^\infty$$

$$= \frac{1}{p+ia} = \frac{p-ia}{p^2+a^2}$$

R e (*p*+*ia*)>0

$$= \frac{p}{p^2+a^2} - i\frac{a}{p^2+a^2}$$