The Error Function and Stirling's Formula

The Error Function:

The curve of the Gaussian function $y = e^{-x^2}$ is called the bell-shaped graph. The error function is defined as the area under part of this curve:

1. $$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt .$$

There are other definitions of error functions. These are closely related integrals to the above one.

2. a) The normal or Gaussian distribution function.

$$P(-\infty, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt = \frac{1}{2} + \frac{1}{2} erf(\frac{x}{\sqrt{2}})$$

Proof: Put $t = \sqrt{2}u$ and proceed, you might reach a step of

$P(-\infty, x) = P(-\infty, 0) + P(0, x)$, where $$P(0, x) = \frac{1}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

Here you can prove that $P(0, x) = \frac{1}{2} erf(\frac{x}{\sqrt{2}})$. This can be done by using the definition of error function in (1).

Now you need to find $P(-\infty, 0) = \frac{I}{\sqrt{\pi}}$ where $I = \int_{-\infty}^{0} e^{-u^2} du$. To find this integral you have to put *u*=*x* first, then *u*= *y* and multiply the two resulting integrals. Make the change of variables to polar coordinate you get

$$I^2 = \int_{-\infty}^{0} e^{-r^2} r dr \int_{0}^{\pi/2} d\theta$$

From this latter integral you get

$$I = \frac{\sqrt{\pi}}{2} \text{ and } \Rightarrow P(-\infty, 0) = \frac{1}{2}.$$

$$\boxed{\therefore P(-\infty, x) = \frac{1}{2} + \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right)}$$

Q. E. D.

2.b $$P(0, x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-t^2/2} dt = \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right)$$

(as proved earlier in 2.a).

3. **a) The complementary error function**

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt = 1 - erf(x)$$

Proof: Put $t^2 = u$ and use the definition (1) of error function and the definition of $\Gamma(1/2)$.

3. **b)** $$erfc\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$$

Proof: (This will be left as an exercise for the students).

4. $$erf(x) = 2P(0, x\sqrt{2}) = 2P(-\infty, x\sqrt{2}) - 1$$

Proof: (The student may take this as another exercise).

Some useful properties of error function:

1. $erf(-x) = -erf(x)$ **This means that the error function is an odd function.**
[solve problem 3 in section 9 chapter 11]

2. $erf(\infty) = 1$. **[This can be shown by putting the limit x equals ∞ and using the definition of Γ(1/2)].**

3. $erf(x) = \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3} + \frac{x^5}{5.2!} - ....)$ **when $|x| << 1$**

Stirling's Formula:

This is an approximated formula for the factorial or Gamma function that can be used to simplify formulas involving factorials.

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

OR

$$\Gamma(p+1) \sim p^p e^{-p} \sqrt{2\pi p}$$

The sign ~, here, means asymptotic to.

Derivation of Stirling's formula:

Starting with $\Gamma(p+1) = p! = \int_0^\infty x^p e^{-x} dx$ **and rewrite the integral as**

$$\Gamma(p+1) = p! = \int_0^\infty e^{p\ell n x - x} dx.$$

Change the variable $x = p + y\sqrt{p}$ **. The limits are** $y = -\sqrt{p}$ **for *x*→0 and *y*→∞ for *x*→∞.**

$$\Gamma(p+1) = p! = \int_{-\sqrt{p}}^\infty e^{p\ell n(p+y\sqrt{p}) - p - y\sqrt{p}} \sqrt{p} dy$$

Firstly we can write

$$\ell n(p + y\sqrt{p}) = \ell n[p(1 + \frac{y}{\sqrt{p}})] = \ell np + \ell n(1 + \frac{y}{\sqrt{p}})$$

For large *p*, the second logarithm on the right hand side of the last equation can be expanded as follows:

$$\ell n(1 + \frac{y}{\sqrt{p}}) = \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \bullet\bullet\bullet\bullet$$

[Hint: The binomial expansion used is:

$$\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \bullet\bullet\bullet\bullet \text{ for } -1 < x \le 1].$$

By substituting the expanded logarithm into the last integral, then we get:

$$\Gamma(p+1) = p! \sim \int_{-\sqrt{p}}^\infty e^{p\ell np + (\frac{py}{\sqrt{p}} - \frac{py^2}{2p}) - p - y\sqrt{p}} \sqrt{p} dy$$

$$\Gamma(p+1) = p! \sim \sqrt{p} e^{p\ell np - p} \int_{-\sqrt{p}}^\infty e^{-\frac{y^2}{2}} dy$$

The integral and its cofactor can be rewritten as:

$$\Gamma(p+1) = p! \sim \sqrt{p}\, p^p e^{-p} \left[\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy - \int_{-\infty}^{-\sqrt{p}} e^{-\frac{y^2}{2}} dy \right].$$

Exercise: **Show that** $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}$

Using the last result mentioned above we get:

$$\Gamma(p+1) = p! \sim \sqrt{2\pi p}\, p^p e^{-p} - \sqrt{p}\, p^p e^{-p} \int_{-\infty}^{-\sqrt{p}} e^{-\frac{y^2}{2}} dy$$

The integral in the last term tends to zero as $p \to -\infty$

$$\therefore \Gamma(p+1) = p! \sim p^p e^{-p} \sqrt{2\pi p}$$

Q.E.D

(This Stirling's formula is a good approximation for large ***p*****)**

Note: When higher terms in the previous expansion of $\ell n(1+\frac{y}{\sqrt{p}}) = \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \bullet\bullet\bullet\bullet$ **are considered, a better asymptotic expansion for the Gamma or factorial function can be obtained.** ***i.e.***

$$\therefore \Gamma(p+1) = p! \sim p^p e^{-p} \sqrt{2\pi p} \left(1 + \frac{1}{12p} + \frac{1}{288p^2} + \cdots\right)$$

(The student can try to prove that at home).