Beta functions

Definitions:

i. $$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$$, *p>0*, *q>0*.

ii. $$B(p,q)=B(q,p)$$

Proof: Put *x* = 1- *y* in (*i*) and proceed.

iii. $$B(p,q)=\frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}dy$$

Proof: Put $x=\frac{y}{a}$ in (*i*) and proceed.

iv. $$B(p,q)=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta$$

Proof: Put $x=\sin^2\theta$ in (*i*) and proceed.

v. $$B(p,q)=\int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy$$

Proof: Put $x=\frac{y}{1+y}$ and proceed.

The relation between the Beta and Gamma functions:

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Proof: **Put t = y^2 for Γ (*p*) and *t* = x^2 for Γ(*q*) and take the product of both functions.[Hint: make the change of variables to polar coordinate].**

Example: Find the integral $I = \int_0^\infty \frac{x^3}{(1+x)^5} dx$ **.**

Solution: This is like $B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}} dy$

Here you need to get the values of *p* and *q* and then use the relation $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q}$ **to find the final answer.**

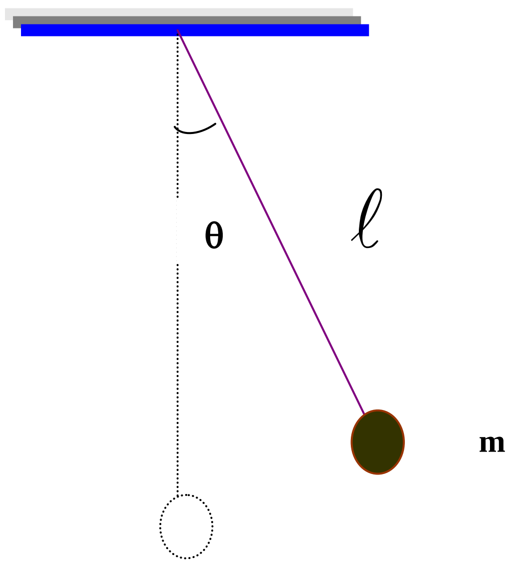
Physical Applications:

(The Simple Pendulum)

The equation of motion of simple pendulum can be developed using the Lagrangian techniques. However the Lagrangian L is defined by

L= T - V.

Where *T* and *V* are the kinetic and the potential energies of the mass m, respectively.



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\ell \dot{\theta})^2 ,$$

where $v = \ell \omega = \ell \dot{\theta}$.

The potential energy of the mass m when it is at an angle θ**:**

$$V = -mg\ell \cos\theta .$$

From the above three equations we get:

$$\therefore L = \frac{1}{2} m (\ell \dot{\theta})^2 + mg\ell \cos\theta .$$

From the general Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

and by using the last obtained Lagrangian we get:

$$\frac{d}{dt}(m\ell^2 \dot{\theta}) - \frac{\partial}{\partial \theta}(mg\ell \cos\theta) = 0$$

$$m\ell^2 \ddot{\theta} + mg\ell \sin\theta = 0$$

$$\therefore \ddot{\theta} = -\frac{g}{\ell} \sin\theta$$

Now we are seeking a solution to this DE.

Case (*i*) : (Approximate Solution)

For small oscillations (*i.e* θ is small) ⇒ $\sin\theta \approx \theta$

⇒
$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

The solution to this DE is either $\theta = \sin\omega t$ **or** $\theta = \cos\omega t$**.**
By considering the first solution and taking its second derivative $\ddot{\theta} = -\omega^2\theta$ **and substituting this into the last DE we get:**

$$-\omega^2\theta = -\frac{g}{\ell}\theta \quad \Leftrightarrow \quad \omega^2 = \frac{g}{\ell}$$

But $\omega = \dfrac{2\pi}{T}$**.**

Thus $T = 2\pi\sqrt{\dfrac{\ell}{g}}$ **is the approximate period when** θ **is small.**

Case (*ii*): (Exact solution)(for any θ)

Multiply both sides of the DE $\ddot{\theta} = -\dfrac{g}{\ell}\sin\theta$ **by** $\dot{\theta}$ **to get:**

$$\dot{\theta}\frac{d\dot{\theta}}{dt} = -\frac{g}{\ell}\sin\theta\frac{d\theta}{dt}$$

$$\dot{\theta}\, d\dot{\theta} = -\frac{g}{\ell}\sin\theta d\theta$$

Integrate both sides to obtain:

$\dfrac{(\dot{\theta})^2}{2} = \dfrac{g}{\ell}\cos\theta + C,$ **where *C* is constant.**

***C* = 0 if** $\theta = \dfrac{\pi}{2}$

$$\boxed{\frac{(\dot{\theta})^2}{2} = \frac{g}{\ell}\cos\theta}$$

$$\therefore \frac{d\theta}{dt} = \sqrt{\frac{2g}{\ell}}(\cos\theta)^{\frac{1}{2}}$$

$$(\cos\theta)^{-\frac{1}{2}} d\theta = \sqrt{\frac{2g}{\ell}}dt$$

$\begin{cases} \theta = 0 \Rightarrow \theta = \dfrac{\pi}{2} \\ t = 0 \Rightarrow t = \dfrac{T}{4} \end{cases}$ **, for one-quarter, we will have**:

$$\int_0^{\pi/2} (\cos\theta)^{-\frac{1}{2}} d\theta = \sqrt{\frac{2g}{\ell}} \int_0^{T/4} dt$$

$$\Rightarrow \quad \boxed{T = 4\sqrt{\frac{\ell}{2g}} \int_0^{\pi/2} (\cos\theta)^{-\frac{1}{2}} d\theta}$$

Recalling that $B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} d\theta$ **and**

Comparing with the integral in the last form of T, we should have

***2p –1* = 0 ⇒ *p*= ½ and *2q –1* =- ½ ⇒ *q*= ¼.**

Thus we need to evaluate B (½, ¼)

$$B(\frac{1}{2}, \frac{1}{4}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

$$\Rightarrow \quad \therefore T = 2\sqrt{\frac{\ell}{2g}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

(This last answer represents an exact form for T).