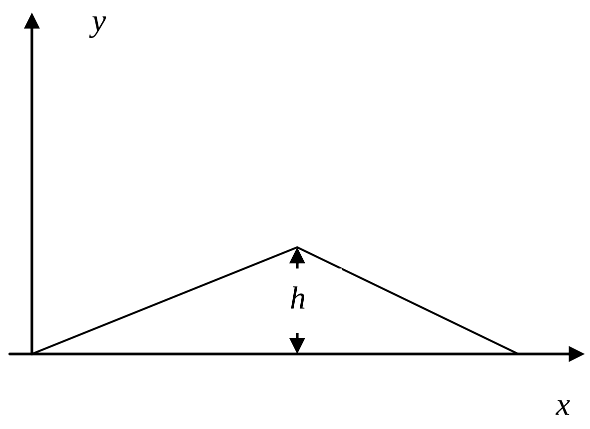
The wave equation

The vibrating string problem:

Problem 1:

A string that is tightly stretched and its ends are fastened to support at $x = 0$ and at the $x = l$. The string is caused to vibrate such that its vertical displacement *y* from its equilibrium position is assumed to be always very small and the slope $\frac{\partial y}{\partial x}$ of the string at any point at any time is also small. Also suppose the string is started vibrating by pulling it aside a small distance h at the center and let it goes (i.e. plucking the string). Find the solution (the displacement y) to this problem.



Solution:

It must be noted that

(a) the vertical displacement, y, depends on x and t,

(b) the length of the string is the same as the distance between the supports,

(c) the string must be stretched a little as it vibrates out of its equilibrium position.

Under the above assumptions, the displacement y satisfies the 1-D wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \ ,$$

where $v(=\sqrt{\frac{T}{\rho}})$ is a constant (called wave velocity) that depends on both the tension in the string, T, and the linear density of the string, ρ.

The initial conditions (I.C's): Plucking process implies that

$$y\,(x,\,0) = y_0 \quad \text{and } y_0 = f(x) = \frac{h}{\ell/2}x \Rightarrow y_0 = \frac{2h}{\ell}x$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = v_0 = 0 \, .$$

The boundary conditions (B.C's): $y\,(0,\,t) = y\,(\ell\,,\,t) = 0$

The separation of variables method is adopted and assumed a solution

$y = X(x)\,T\,(t)$.

Substitute the assumed solution into the PDE to get:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2T}\frac{d^2T}{dt^2} = -k^2$$

$\Rightarrow \quad X'' + k^2X = 0$ (This is the space part of DE). Its solution is:

$$X = \begin{cases} \sin kx \\ \cos kx \end{cases}$$

Also we will have the time part (DE) is equal to $-k^2$ such that:

$$\ddot{T} + k^2v^2T = 0 \, .$$

The latter DE has the solution of type $T = \begin{cases} \sin \omega t \\ \cos \omega t \end{cases}$

Recalling that $k = \frac{\omega}{v}$, $\omega = 2\pi\nu$ and $v = \lambda\nu$ from wave phenomenon.

[Note: Again the choice of $-k^2$ (and not k) is made because (a) the solution must describe vibrations which are represented by sines and cosines and not by exponentials, (b) the boundary conditions for real k must be satisfied].

The basic solution to the full PDE becomes:

$$y = \begin{cases} \sin kx \sin \omega t \\ \sin kx \cos \omega t \\ \cos kx \sin \omega t \\ \cos kx \cos \omega t \end{cases}$$

B.C's ($x = 0$ and $x = \ell$) imply that $y = 0$ for these values of x and all values of t.

Only $\sin kx$ terms are survived. Also $\sin k\ell = 0 \Rightarrow k = \frac{n\pi}{\ell}$.

Thus the solution becomes

$$y = \begin{cases} \sin \frac{n\pi}{\ell} x \sin \frac{n\pi v}{\ell} t \\ \sin \frac{n\pi}{\ell} x \cos \frac{n\pi v}{\ell} t \end{cases}$$

The last choice of solutions depends on the initial conditions.

Exercise: Suppose that the string of previous problem is started vibrating by plucking (pulling it aside a small distance h at the center and letting it goes); i.e. at $t = 0$, $y_0 = f(x)$ and the velocity of points on the string $\frac{\partial y}{\partial t} = 0$ .Find the solution (the displacement y)

[Hint: do not confuse $\frac{\partial y}{\partial t}$ with the wave velocity v, there is no relation between them].

Answer: $y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} \cos \frac{n\pi v t}{\ell}$. (Solve problem 4.1)

Exercise: Suppose that the string of the previous problem is started by hitting it (like a piano string, for example); *i.e* the initial conditions are $y = 0$ at $t = 0$, and $\frac{\partial y}{\partial t} = V(x)$ at $t = 0$.Find the solution (the displacement y). (Solve problem 4.5 - 4.8)