The diffusion or heat flow equation

The heat flow in a slab or bar problem:

Problem 1:

Consider the flow of heat through a slab of thickness l with insulated walls such that the heat flow will be just in the x-axis. Suppose the bar has initially a steady-state temperature distribution with the $x = 0$ wall at 0° and the $x = l$ wall at 100° . From t = 0 on, let the $x = l$ wall (as well as the $x = 0$ wall) be held at 0°. Find the temperature at any x (in the slab) at any later time.

Solution:

This is a one dimensional space dependent (along x-axis) problem with time dependent. The temperature distribution function $u(x, t)$ is the non-steady temperature in a region with no heat sources.

This problem will b solved using the heat flow PD Equation:

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

α^2 is the characteristic constant of the material through which heat is flowing.

Assume a solution to this PDF of the form:

$u = F(x, y, z)T(t)$

F is the three dimensional space dependent part of u which will be reduced to $F(x)$ in our case.

T is the time-dependent part of u.

Substitute the assumed solution into the PDE to get:

$$T\nabla^2 F = \frac{1}{\alpha^2} F \frac{dT}{dt}$$

$$\Rightarrow \qquad \frac{\nabla^2 F}{F} = \frac{1}{\alpha^2 T}\frac{dT}{dt}$$

(This is the PDE as separated to a space and time parts).

The space part of DE is set equal to $-k^2$ and we have

$$\frac{\nabla^2 F}{F} = -k^2 \Rightarrow \qquad \nabla^2 F + k^2 F = 0$$

Also we will have the time part (DE) is equal to $-k^2$ such that:

$$\frac{1}{\alpha^2 T}\frac{dT}{dt} = -k^2 \Rightarrow \qquad \frac{dT}{dt} = -k^2\alpha^2 T$$

The latter DE has the solution of type $T = e^{-k^2\alpha^2 t}$

[Note: $-k^2$ was chosen to meet the physics of the problem. As time t increases the temperature of the body may decrease to zero].

Since the space part of our problem is restricted to one dimension (x-direction), the space part of DE becomes $\frac{d^2F}{dx^2} + k^2 F = 0$ and its assumed solution is: $u = F(x)\, T(t)$.

The initial conditions (I.C's): Implies that $t = 0$, such that

$$u(x, 0) = u_0(x)$$

$$u_0(0) = 0$$

$$u_0(\ell) = 100$$

The Boundary conditions (B.C's):

$$u(0, t) = u(\ell, t) = 0$$

The initial steady-state temperature distribution $u_0(x)$ must be found at first.

Here $u_0(x)$ satisfies Laplace's equation; i.e. $\nabla^2 u_0 = 0$

In 1-D: $\frac{d^2u_0}{dx^2} = 0$

The solution to this DE is $u_0 = ax + b$ where a and b are constants which can be found from the given initial conditions.

Since at $x = 0 \Rightarrow u_0(0) = 0$ and $x = \ell \Rightarrow u_0(\ell) = 100$,

then for $x = 0 \Rightarrow b = 0$ and for $x = \ell \Rightarrow a = \frac{100}{\ell}$

$\Rightarrow$

$$\boxed{u_0(x) = \frac{100}{\ell}x}$$

From t = 0 on, $u(x, t)$ satisfies the heat flow DE, such that,

$u = F(x)\,T(t)$, where the space part $\frac{d^2F}{dx^2} + k^2F = 0$ has the solution:

$F(x) = A\sin kx + B\cos kx$.

Here A and B are constants needed to be determined.

Since B.C requires that at $x = 0$, $u(0, t) = 0$ this implies that $B = 0$.

The solution to the heat flow DE becomes:

$$u(x,t) = A\sin kx e^{-k^2\alpha^2 t}$$

The B.C at $x = \ell$ gives $u(\ell, t) = 0$ and this implies that $\sin k\ell = 0$

$\Rightarrow \quad k = \frac{n\pi}{\ell}$.

$$\Rightarrow u_n(x,t) = A_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}$$

The linear combination of n solutions is the suitable solution to this problem, i.e. $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$$\Rightarrow \qquad u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{\ell}) e^{-(\frac{n\pi\alpha}{\ell})^2 t}$$

At $t = 0$, we found that $u(x, 0) = u_0(x) = \frac{100}{\ell}x$. When we substitute this into the last solution we get $\frac{100}{\ell}x = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{\ell})$.

This last result will allow us to obtain the coefficients A_n from the Fourier sine series for $\frac{100}{\ell}x$ on (0, ℓ), as follows:

$$A_n = \frac{2}{\ell}\int_0^{\ell} \frac{100}{\ell} x \sin(\frac{n\pi x}{\ell}) dx = \frac{200}{\ell^2}\int_0^{\ell} x \sin(\frac{n\pi x}{\ell}) dx$$

Using the identity $\int u dv = uv - \int v du$

$u = v,\ du = dx$,

$dv = \sin(\frac{n\pi x}{\ell}) dx$, $v = -\frac{\ell}{n\pi}\cos(\frac{n\pi x}{\ell})$

$$A_n = -\frac{\ell x}{n\pi}\cos(\frac{n\pi x}{\ell})\Big|_0^{\ell} + \frac{\ell}{n\pi}\int_0^{\ell} \cos(\frac{n\pi x}{\ell}) dx$$

$$A_n = \frac{200}{\pi}\frac{(-1)^{n-1}}{n}$$

The final solution is:

$$u(x,t) = \frac{200}{\pi}[e^{-(\frac{\pi\alpha}{\ell})^2 t} \sin(\frac{\pi x}{\ell}) - \frac{1}{2}e^{-(\frac{2\pi\alpha}{\ell})^2 t} \sin(\frac{2\pi x}{\ell}) + \frac{1}{3}e^{-(\frac{3\pi\alpha}{\ell})^2 t} \sin(\frac{3\pi x}{\ell}) - \bullet\bullet\bullet]$$

Exercises:

1) Suppose that the final temperatures of the faces in the previous problem are two different constant values (different from zero).

[Hint: If u_f is the linear function representing the correct final steady state, then the solution will be

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{\ell}) e^{-(\frac{n\pi\alpha}{\ell})^2 t} + u_f \ .$$

Then for t = 0, $$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{\ell}) + u_f \ ].$$

2) Suppose that the faces of the slab in the previous problem are insulated where no heat flows in or out of the slab. This will be true if normal derivative $\frac{\partial u}{\partial n}$ of the temperature is zero at the boundary (Neumann condition), i.e. $\frac{\partial u}{\partial x} = 0$ at $x = 0$ and $\frac{\partial u}{\partial x} = 0$ at $x = \ell$. This means that the appropriate solution may be $u(x,t) \propto e^{-k^2\alpha^2 t} \cos kx$ (solve problem 3.7).