Steady-State Temperature in a Sphere

Laplace's equation in spherical coordinates

Problem 1:

Find the steady-state temperature inside a sphere of radius $r = 1$ when the surface of the upper half is held at 100° and the surface of the lower half at 0°.

Solution:

a) Since there is no source of heat is available inside the sphere, the temperature u satisfies Laplace's equation.

b) The symmetry of the problem implies the use of spherical coordinates.

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right]$$

In spherical coordinates:

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$$

$$\partial x_1 = \partial r \qquad \partial x_2 = \partial\theta \qquad \partial x_3 = \partial\phi$$

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\phi}\right)\right]$$

But

$$\nabla^2 u = 0$$

$$\therefore\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial u^2}{\partial^2\phi}=0$$

Try a solution of type $u(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$

Substitute this solution into the PDF and multiply both sides by

$\frac{r^2\sin^2\theta}{R\Theta\Phi}$ to get:

$$\frac{\sin^2\theta}{R}\frac{d}{dr}(r^2\frac{dR}{dr})+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0$$

Put $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2$ (The solution to this DE is $\Phi=C\sin\phi+D\cos\phi$)

After substituting the latter DF into the PDF and dividing by $sin^2\theta$, the PDF becomes:

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr})+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})-\frac{m}{\sin^2\theta}=0$$

Now this DF is separable. The radial part of this equation is set equal to a constant k.

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr})=k$$

[Note: It is more suitable to write k as the product of two successive integers (*i.e.* $k=\ell(\ell+1)$)].

$$\frac{d}{dr}(r^2\frac{dR}{dr})=\ell(\ell+1)R$$

The last differential equation has the form:

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\ell(\ell+1)R=0$$

Assume a solution: $R = r^n$ and substitute it into the DE to get:

$$n(n-1)\,r^n + 2nr^n - \ell(\ell+1)\,r^n=0$$

Equating the coefficients of r^n in this equation

$$n^2 + n - \ell(\ell+1) =0$$

Thus n has two roots:

$$n = \ell \quad \text{and} \quad n = -(\ell + 1)$$

The general solution to the radial equation is a linear combination of two solutions, *i.e.* $R = Ar^{\ell} + Br^{-\ell-1}$

Remainder of DF:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + [\ell(\ell+1) - \frac{m}{\sin^2\theta}]\Theta = 0$$

[This is the associated Legendre DE which hives the common solution of associated Legendre polynomial, i.e. $\Theta = P_{\ell}^{m}(\cos\theta)$ (see problem 10.2 in chapter 12).

Thus the general solution ($u = R\Theta\Phi$) becomes:

$$u = (Ar^{\ell} + Br^{-\ell-1})(C\sin m\phi + D\cos m\phi)P_{\ell}^{m}(\cos\theta)$$

Notes:

1) Since we are interested to find the temperature inside the sphere, we have to consider $B = 0$, because $r^{-\ell-1}$ goes to infinity at the origin ($r = 0$).
2) The problem has azimuthal symmetry i.e. u is independent of ϕ (as ϕ changes u is constant). This implies that $u = D\cos m\phi$ with $m = 0$ and $\cos m\phi$.

$\Rightarrow u = D$ [This can be justified for the given B.C where the top of the sphere is at 100° and the bottom of the sphere at 0°].

For $m \neq 0$

$$u_{\ell} = A' r^{\ell} \cos m\phi P_{\ell}^{m}(\cos\theta)$$

[Note: The spherical harmonic $Y_{\ell}^{m}(\theta,\phi)$ is related to the associated Legendre polynomial can be expressed

as: $Y_{\ell}^{m}(\theta,\phi)\sqrt{\dfrac{2\pi}{2\ell+}\dfrac{(\ell-m)!}{(\ell+m)!}} = P_{\ell}^{m}(\cos\theta)\cos m\phi$].

Here, in this problem we have *m = 0* and the solution is reduced to: $u_\ell = A' r^\ell P_\ell^m(\cos\theta)$

$\Rightarrow u = \sum_\ell u_\ell$.

And $u = \sum_{\ell=0}^{\infty} A'_\ell r^\ell P_\ell(\cos\theta)$

The coefficients A'_ℓ can be determined using the given temperatures when $r = 1$.

$$u(r=1,\theta) = \begin{cases} 100 & 0 < \theta < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \theta < \pi \end{cases}$$

$$u(r=1,\theta) = \sum_{\ell=0}^{\infty} A'_\ell P_\ell(x)$$, (where $x = \cos\theta$)

Also we have $u(r=1,\theta) = 100 f(x)$, where

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

$$\therefore \quad \sum_{\ell=0}^{\infty} A'_\ell P_\ell(x) = 100 f(x)$$

$$A'_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} u(x) P_\ell(x)\,dx$$

$$\Rightarrow \quad A'_\ell = \frac{2\ell+1}{2}[100\int_{0}^{1} P_\ell(x)\,dx]$$

Thus $A_0 = \frac{100}{2}\int_0^1 dx = \frac{100}{2}$; $A_1 = \frac{300}{2}\int_0^1 x\,dx = \frac{300}{4}$ and $A_3 = -\frac{700}{16}$ ...ect.

$$\therefore u(1,\theta) = 100[\frac{1}{2}P_\circ(\cos\theta) + \frac{3}{4}P_1(\cos\theta) - \frac{7}{16}P_3(\cos\theta) + \bullet\bullet\bullet].$$

(Solve problem 7.13)