Steady-state temperature in a cylinder

Problem 3:

Find the steady-state temperature distribution **u** in a semi-infinite solid cylinder of radius $r = \mathbf{1}$ if the base is held at 100° C and the curved sides at 0° C.

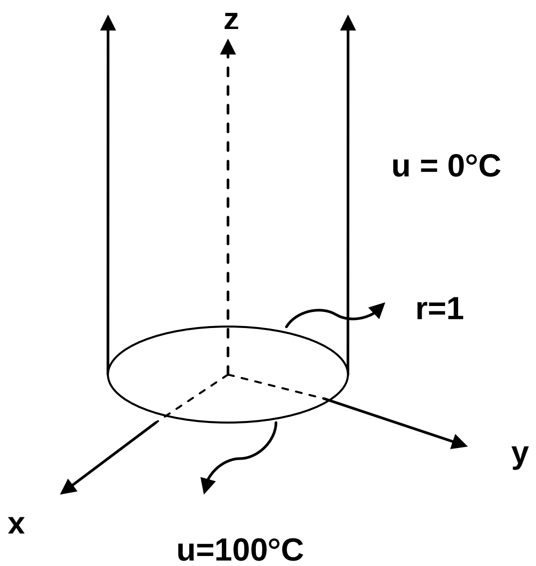
Solution:

Boundary conditions (B.C's):

a) $u \to 0$ at $z \to \infty$

b) $u = 0$ for $r = 1$

c) u = 100° C (at different θ around the base for $z = 0$)



B.C's imply that it is suitable to use cylindrical coordinates to solve the problem.

Lap lace's equation in cylindrical coordinates:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Separation of variables method:

Assume the solution $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$

Substitute this solution into the Lap lace's equation:

$$\frac{Z\Theta}{r}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{RZ}{r^2}\frac{d^2\Theta}{d\theta^2} + R\Theta\frac{d^2 Z}{dz^2} = 0$$

Divide by $R\Theta Z$

$$\frac{1}{Rr}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{1}{\Theta r^2}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

$$\therefore \frac{1}{Z}\frac{d^2 Z}{dz^2} = k^2 \qquad (k > 0)$$

$$\therefore \frac{1}{Rr}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{1}{\Theta r^2}\frac{d^2\Theta}{d\theta^2} = -k^2$$

[Hint: None of the terms on the left hand side of the equation is a constant, because both terms contain r**].**

Note: For a term to be constant;

a) It must be a function of one variable only.

b) And that variable does not appear elsewhere in the equation.

Thus $\frac{d^2 Z}{dz^2} - k^2 Z = 0$

$Z = Ae^{kz} + Be^{-kz}$

$$\therefore \frac{1}{Rr}\frac{d}{dr}(r\frac{dR}{dr})+\frac{1}{\Theta r^2}\frac{d^2\Theta}{d\theta^2}+k^2=0$$

To make the separation of variables again to this equation:

Firstly multiply both sides by r^2

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr})+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+k^2r^2=0$$

Secondly, separate the Θ- equation.

The second term contains the variable θ only, so

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-n^2$$

[Note: $-n^2$ is chosen because

a) Θ is periodic; where the variable θ is the same as $\theta + 2m\pi$

b) There is one physical point and one temperature whatever the value of m is].

$$\therefore \frac{d^2\Theta}{d\theta^2}+n^2\Theta=0$$

$$\therefore \Theta = C\sin n\theta + D\cos n\theta$$

Finally, the r equation is

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr})-n^2+k^2r^2=0$$

OR

$$r\frac{d}{dr}(r\frac{dR}{dr})+(k^2r^2-n^2)R=0$$

Since **Bessel** differential equation is

$x^2 y'' + xy' + (x^2 - p^2)y = 0$ was rewritten

as $x(xy')' + (x^2 - p^2)y = 0$.

Recalling the concept of replacing x by ax

$$x(xy')' + (a^2 x^2 - p^2)y = 0$$

This has a solution $J_p\,(ax)$.

If we have a_m with (m = 1, 2, 3,...) as the zeros of $J_p\,(ax)$, then

$\sqrt{x} J_p(a_m x)$ are orthogonal on (0, 1) interval.

Put $x = r$, $a_m = k_m$ and $p = n$.

Solutions are $J_n\,(k_m\, r)$ and not $N_n\,(k_m\, r)$ because the base of the cylinder contains the origin (*i.e.* $N_n\,(k_m r) \rightarrow$ infinity at $r = 0$).

$\therefore R_n(r) = F_m\, J_n\,(k_m r)$

For only one value of n, there are (m= 1,2,3,.....) possible values of k**.** These values of k are the zero of J_n at the particular n**.**

B.C's:

$u = 0$ for $r = 1$

$R(r) = 0$ for $r = 1$

Also the B.C $u \rightarrow 0$ as $z \rightarrow \infty$ implies that $A = 0$

$$\therefore u_m = F_m(C_m \sin n\theta + D_m \cos n\theta)\, J_n(k_m r)\, B_m e^{-k_m z}$$

The B.C $u = 0$ when $r = 1$ for all θ and z (where $\theta = \theta + 2m\pi$) gives

$u = \sum_{m=1}^{\infty} u_m = \sum A'_m \cos n\theta J_n(k_m r) e^{-k_m Z} + B'_m \sin n\theta J_n(k_m r) e^{-k_m Z}$, for a fixed value of n.

B.C: u = 100°C for different θ around the base**.**

This means that at the base of the cylinder u is constant as θ is changing. This means that we have to use $n = 0$ such that Θ = constant = D_m.

$$\therefore u = \sum_{m=1}^{\infty} A'_m J_0(k_m r) e^{-k_m Z}$$, where $A'_m = B_m F_m D_m$.

We need to find the coefficient A'_m.

Use the **B.C** u = 100°C when $z = 0$

$$100 = \sum_{m-1}^{\infty} A'_m J_0(k_m r)$$ **(**This is the Fourier- Bessel series**)**.

The function $u(r, \theta, 0) = 100$ is expanded in a series of Bessel functions.

Multiply Both sides by $rJ_0(k_s r)$ (where s = 1, 2, 3,.....etc).

And integrate from $r = 0$ to $r = 1$ to get:

$$\int_0^1 \sum_{m=1}^{\infty} A'_m \, r J_0(k_s r) J_0(k_m r)\, dr = \int_{\partial}^{1} 100\, r J_0(k_s r)\, dr$$

All terms on L.H.S vanish except the term with $m = s$.

$$A'_s \int_{\partial}^{1} r[J_0(k_s r)]^2 dr = \int_{\partial}^{1} 100 r J_0(k_s r) dr$$

$$\therefore A'_s = \frac{100\int_{\partial}^{1} rJ_0(k_s r)dr}{\int_{\partial}^{1} r[J_0(k_s r)]^2 dr}$$

To find Denominator:

Since $\int_{\partial}^{1} rJ_p(ar)J_p(br)dr = \frac{1}{2}J_{p+1}{}^2(a)$ for $a = b$

$\Rightarrow$ Denominator becomes $\frac{1}{2}J_1^2(k_m)$

To find numerator:

Also since $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$.

$\therefore$ Put $x=R_m r$ to get:

$$\frac{1}{k_m}\frac{d}{dr}[k_m rJ_1(k_m r)]dr = k_m rJ_0(k_m r).$$

Integrate to get:

$$\Rightarrow \frac{1}{k_m}\int_0^1 \frac{d}{dr}[rJ_1(k_m r)]dr = k_m \int_0^1 rJ_0(k_m r)dr$$

$$L..H.S : rJ_1(k_m r)\Big|_0^1 = J_1(k_m)$$

$$\therefore \int_0^1 rJ_0(k_m r)dr = \frac{J_1(k_m)}{k_m}$$

$\therefore$ Numerator $= \frac{100 J_1(k_m)}{k_m}$

$$\therefore\ A'_s = \frac{200 J_1(k_m)}{k_m J_1^2(k_m)} = \frac{200}{k_m J_1(k_m)}$$

We have to remember here that k_m is the zero of J_o and not J_1.

So (a) Either we need to find the values of J_1 (or $J'_o =- J_1$) (from tables of Bessel functions) at the zeros of J_o.

Or,

(b) we can, at first, find the values of k_m (zeros of J_o) and then interpolate in a J_1 table to find the values of $J_1(k_m)$.

$\therefore$ The final solution becomes:

$$u = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{200}{k_m J_1(k_m)} J_o(k_m r) e^{-k_m z}$$

Exercise:

Suppose that the given temperature of the base of the cylinder on the previous example is $f(r, \theta)$. Find the solution in such case.

Answer:

$$u = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} J_n(k_{mn} r)(C_{mn} \sin n\theta + D_{mn} \cos n\theta) e^{-k_{mn} z}$$

At $z = 0$ we need $u = f(r,\theta)$.