Exercise: If at the bottom of the plate in the previous problem (problem 1) $T \approx f(x)$ instead of 100°C we repeat the same method by expanding $f(x)$ in a Fourier sine series and substitute the coefficient into $T(x, y)$ equation.

Problem 2:

The problem of finite plate (consider the height ℓ =30 cm) with the top edge at T =0°. All other dimensions and temperatures are same as before.

As T= 0 when y= 30 cm, then $0 = Ce^{30k} + De^{-30k}$

$$\frac{C}{D} = \frac{e^{-30k}}{e^{30k}} = \frac{-\frac{1}{2}e^{-30k}}{-\frac{1}{2}e^{30k}}$$

$$\therefore T = \left[-\frac{1}{2}e^{-k(30-y)} + \frac{1}{2}e^{k(30-y)}\right]x\,(A \sin kx + B \cos kx)$$

$$\sinh k(30\text{-}y) = \frac{1}{2}e^{k(30-y)} - \frac{1}{2}e^{-k(30-y)}$$

As T= 0 when x= 0 $\Rightarrow$A= 0

$$T(-x, y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{10}(30-y)\frac{\sin n\pi x}{10} \quad \text{where } k=\frac{n\pi}{10}$$

Each term of this series is zero on the three $T = 0$ sides of the plate.

When $y = 0$, T= 100°

$$100 = \sum_{n=1}^{\infty} B_n \sinh(3n\pi)\sin\frac{n\pi x}{10} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

Where $b_n = B_n \sinh(3n\pi)$

Since b_n was found previously as $b_n = \frac{400}{n\pi}$ when n is odd

$= 0$ when n is even.

$$\therefore B_n = \frac{400}{n\pi \sinh(3n\pi)} \Rightarrow T = \sum_{n \equiv odd} \frac{400}{n\pi \sinh 3n\pi} \sinh \frac{n\pi}{10}(30 - y) \sin \frac{n\pi x}{10}$$

Important remarks:

The reason behind choosing $-k^2$ for the space ($AF=\lambda F$) eq:

(1) put $\lambda = 0$

$$\frac{d^2F}{dx^2} + k^2 F = 0 \Rightarrow \frac{d^2F}{dx^2} \Rightarrow F(x) = C + Dx$$

But $F(0) = F(\ell) = 0 \Rightarrow C = D = 0$

$\therefore F(x) = 0$ No solution $\Rightarrow$zero is not an eigenvalue

(2) Put $\lambda < 0$ $\lambda == k^2$

$$\frac{d^2F}{dx^2} = k^2 F \Rightarrow F(x) = C\cosh kx + D\sinh kx$$
$$F(0) = 0 \Rightarrow = 0$$
$$F(\ell) = 0 \Rightarrow D\sinh k\ell = 0$$

But $\sinh k\ell \neq 0$

$\therefore D = 0$ and the solution is $F(x) = C\cosh kx$

$3) \lambda = -k^2 \Rightarrow \lambda = ik$

$$\therefore F(x) = Ce^{ikx} + De^{-ikx}$$

B.C's: $F(0) = 0$ at $x = 0$

$0 = C + D \Rightarrow C = -D$

$F(\ell)=0$ at $x=\ell$

$$0=Ce^{ik\ell}+De^{-ik\ell}$$
$$0=C(e^{ik\ell}-e^{-ik\ell})$$
$$0=k'\sin k\ell$$
$$k\ell=n\pi$$
$$k=\frac{n\pi}{\ell}$$

The square of k^2 is $(\frac{n\pi}{\ell})^2 \Rightarrow (\frac{\pi}{\ell})^2,(\frac{2\pi}{\ell})^2$ and so on, which are all real and positive.