Partial Differential Equations

Several Kinds of Physical problems that lead to the PDE:

(1) Laplace's equation $\nabla^2 u = 0$

(2) Poisson's equation $\nabla^2 u = f(x, y, z)$

(3) The diffusion or heat flow equation $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$

(4) The wave equation $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

(5) Helmholtz equation $\nabla^2 F + k^2 F = 0$

We will adopt the concept of separation of variables to simplify the problem under study.

Required information:

$$\vec{\nabla} u = \frac{\hat{e}_1}{h_1}\frac{\partial u}{\partial x_1} + \frac{\hat{e}_2}{h_2}\frac{\partial u}{\partial x_2} + \frac{\hat{e}_3}{h_3}\frac{\partial u}{\partial x_3}$$

$$\vec{\nabla} \bullet \vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_2}(h_1 h_3 v_2) + \frac{\partial}{\partial x_3}(h_1 h_2 v_3)\right]$$

$$\vec{\nabla} \times \vec{v} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}$$

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right]$$

Cylindrical coordinates:

$$h_1 = 1 \quad h_2 = r \quad h_3 = 1$$

$\hat{e}_1 = r$ $\qquad$ $\hat{e}_2 = \hat{\theta}$ $\qquad$ $\hat{e}_3 = \hat{k}$

$\partial x_1 = \partial r$ $\qquad$ $\partial x_2 = \partial\theta$ $\qquad$ $\partial r_3 = \partial z$

Spherical coordinates:

$$h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$$

$\hat{e}_1 = \hat{r}$ $\qquad$ $\hat{e}_2 = \hat{\theta}$ $\qquad$ $\hat{e}_3 = \hat{\phi}$

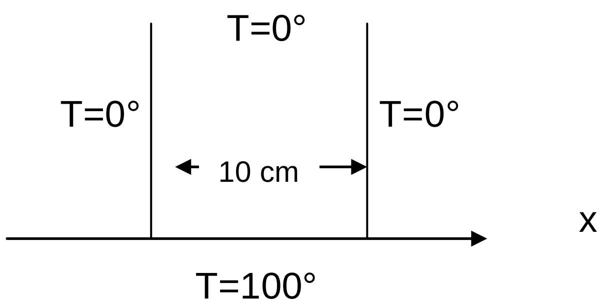
$\partial x_1 = \partial r$ $\qquad$ $\partial r_2 = \partial\theta$ $\qquad$ $\partial r_3 = \partial\phi$

Laplace`s Equation:

Steady –State Temperature in a Rectangular Plate:

Problem 1:

A long rectangular metal plate has its two long sides and the far end at 0° and the base at 100°, as shown. The width of the plate is 10 cm. Find the steady –state temperature distribution inside the plate.



Solution:

This problem is called the semi-infinite plate. This is obtained by simplifying the problem in making the assumptions that:

1- The plate length $\ell \gg$ its width W

2- $\ell \to \infty$ in y-direction.

The last assumption is good for obtaining temp not too near the far end.

Boundary conditions (B.C's):

1) $T \longrightarrow 0$ when y $\longrightarrow \infty$

2) $T = 0$ when $x = 0$

3) $T = 0$ when $x = 10$ cm

4) $T = 100°$ when y $= 0$

Which equation the temperature distribution function $T\ (x, y)$ must satisfy? And where?

$T\ (x, y)$ must satisfy Laplace's equation inside the plate where there are no sources of heat ($i.e. \nabla^2 T = 0$).

Laplace's equation will be written in rectangular coordinate because the boundary of the plate is rectangular.

Thus $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

Try a solution $T\ (x, y) = \mathrm{X}\ (x)\ \mathrm{Y}(y)$

Substitute this solution into the DE to get

$\mathrm{Y}\frac{d^2\mathrm{X}}{dx^2} + \mathrm{X}\frac{d^2\mathrm{Y}}{dy^2} = 0$ (This is ODE instead of PDE because X is only a function of x and Y is only a function of y).

∴We get the identity:

$$\frac{1}{\mathrm{X}}\frac{d^2\mathrm{X}}{dx^2} + \frac{1}{\mathrm{Y}}\frac{d^2\mathrm{Y}}{dy^2} = 0$$

Now, we can start the process of separation of variables. Here, we have an equation of the form: f(x) + g(y) =0. This equation could be true only if both f and g are constants (where x and y are independent variables). This is the basis of separation of variables.

To have the identity equation satisfied, suppose we choose a particular value to x into the first term, the second term must be minus the same chosen value (a constant). While x is constant keep varying y such that the identity is still satisfied. The second term $(-\frac{1}{Y}\frac{d^2Y}{dy^2})$ remains constant as y varies. In the same way we fix y and vary x to make sure that the 1st term is a constant.

$\therefore \frac{1}{X}\frac{d^2X}{dx^2} = -k^2$ and $\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2$

$\therefore k^2$ is called the separation constant (where k ≥ 0)

The X – equation:

$$X = \begin{cases} \sin kx \\ \cos kx \end{cases}$$

The Y-equation:

$$Y = \begin{cases} e^{ky} \\ e^{-ky} \end{cases}$$

$\Rightarrow$

$$\therefore \quad T = X(x)Y(y) = \begin{cases} e^{ky} \sin kx \\ e^{-ky} \sin kx \\ e^{ky} \cos kx \\ e^{-ky} \cos kx \end{cases}$$

OR

X= A $\sin kx$ + B $\cos kx$

&

Y = Ce^{ky} + De^{-ky}

Thus

$T(x, y)$ = (A $\sin kx$ + B $\cos kx$) (C e^{ky} + D e^{-ky})

Where A, B, C and D are arbitrary constants and need to be found by imposing the boundary conditions.

1) If $y \to \infty$ then $T \to 0$. This implies that C= 0 (we are assuming that $k >$ 0) and we are left with

$$T = De^{-ky} (A \sin kx + B \cos kx)$$

$= A'e^{-ky}\sin kx + B'e^{-ky}\cos kx$

where $A' = AD$ & $B' = BD$

2) If x=0 then T =0: this implies $B' = 0$

So we are left with $T = A' e^{-ky} \sin kx$

Also we still have boundary conditions

(T=0 when x= 10 cm & T=100°C when y=0)

3) Since the value of k is required, we can make use of the boundary condition, When x=10cm, we have T=0,

$\sin 10k = 0$

$\Rightarrow k = \frac{n\pi}{10}$ (n=1, 2, 3….)

∴For any integral n, the solution $T_n = A'_n e^{-\frac{n\pi y}{10}} \sin\frac{n\pi x}{10}$

satisfies the given boundary conditions on the three T=0 sides of the plate.

4) Finally we should have T=100 when y=0, this condition is not satisfied by last expression for any n. But a linear combination of such solutions is the required solution.

$$T(x,y) = \sum_n T_n = \sum_{n=1}^{\infty} A'_n e^{-\frac{n\pi y}{10}} \sin\frac{n\pi x}{10}$$

This solution also still meets the other three boundary conditions (*i.e.* T=0 when x=0, x=10 cm &$T\to 0$ when $y\to\infty$).

The last form of the solution will be achieved by making the reasonable choice of coefficients A'_n .

For y=0 we must have T=100°: $T(x,0) = \sum_{n=1}^{\infty} A'_n \sin\frac{n\pi x}{10}$

$$100 = \sum_{n=1}^{\infty} A'_n \sin\frac{n\pi x}{10}$$ (This is just the Fourier sine series of $f(x) = 100$ with $\ell = 10cm$)

$$\therefore A'_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$= \frac{2}{10}\int_0^{10} 100\sin\frac{n\pi x}{10}dx$$

$$= (20)\frac{10}{n\pi}\left(-\cos\frac{n\pi x}{10}\right)\Bigg|_0^{10}$$

$$= -\frac{200}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{400}{n\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$\therefore T(x, y) = \frac{400}{\pi}\left(e^{-\frac{\pi y}{10}}\sin\frac{nx}{10} + \frac{1}{3}e^{-\frac{3\pi y}{10}}\sin\frac{3\pi x}{10} + \bullet\bullet\bullet\bullet\right)$$

Let us find the temperature at the middle of the plate (*i.e.* $x = 5$ cm and $y = 5$ cm)

$$T(5,5) = \frac{400}{\pi}\left(e^{-\frac{\pi}{2}}\sin\frac{\pi}{2} + \frac{1}{3}e^{-\frac{3\pi}{2}}\sin\frac{3\pi}{2} + \bullet\bullet\bullet\right)$$

$$= \frac{400}{\pi}(0.208 - 0.003 + \ldots\ldots) = 26.1^{\circ}$$