Laguerre Functions:

The DE for the Laguerre polynomials is:

$xy'' - (1-x)y' + ny = 0$ **has solutions *y= $L_n(x)$* which *can be obtained from a Rodrigues formula like:***

$$L_n(x) = \frac{1}{n!}e^x \frac{d^n}{dx^n}(x^n e^{-x}).$$

(See problem 22.13).

For n=0 ⇒ $L_o(x)= 1$
For n= 1 ⇒ $L_1(x)= 1 - x$
For n= 2 ⇒ $L_1(x)= 1 - 2x + x^2/2$.
And so on….

The Laguerre polynomials can also be obtained from a generating function like: $\Phi(x,h) = \frac{e^{-xh/(1-h)}}{(1-h)}$.

We can show that: $\frac{e^{-xh/(1-h)}}{(1-h)} = \sum_0^{\infty} h^n L_n(x)$ ***(Solve problem 22.17)***

Some properties of Laguerre polynomials:

1. $\int_0^{\infty} e^{-x} L_n(x) L_m(x) dx = \delta_{nm} = \begin{cases} 0 & if \quad n \neq m \\ 1 & if \quad n = m \end{cases}$.

(Solve problem 22.19).

2. There is a set of recursion relations for Laguerre polynomials:

i) $L'_{n+1}(x) - L'_n(x) + L_n(x) = 0$

ii) $(n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + nL_{n-1}(x) = 0$

iii) $xL'_n(x) - nL_n(x) + nL_{n-1}(x) = 0$

(Solve problem 22.18).

Associated Laguerre polynomials:

These Polynomials can be obtained by taking the derivatives of the above Laguerre polynomials $L_n(x)$. Thus, you can use the relation $L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x)$***. (Solve problem 22.20).***

[Note: $L_n^k(x)$ are used in the theory of the hydrogen atom in Q.M.].

However, the DE for the associated Laguerre polynomials is:

$xy'' - (k+1-x)y' + ny = 0$ **has solutions y =** $L_n^k(x)$**.**
Here, you can also differentiate the Laguerre DE and show that the resulting DE can be satisfied by $L_n^k(x)$***.***
(Solve problem 22.21).

$L_n^k(x)$ ***can also be obtained from the Rodrigues formula:***

$$L_n^k(x) = \frac{x^{-k}e^x}{n!}\frac{d^n}{dx^n}(x^{n+k}e^{-x}).$$

[Note, here that k may be an integer or takes values > -1].

Some properties of associated Laguerre polynomials:

1. $\int_0^\infty x^k e^{-x} L_n^k(x) L_m^k(x)\,dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{(n+k)!}{n!} & if \quad n = m \end{cases}$ ***.***

(Solve problem 22.24, 25).

2. You may find different normalization integral from that in (1) in the theory of hydrogen atom, and as follows:

$$\int_0^\infty x^{k+1} e^{-x}[L_n^k(x)]^2\,dx = (2n+k+1)\frac{(n+k)!}{n!}.$$

(See problems 22.25-27).

3. There are recursion relations for associated Laguerre polynomials:

i) $$(n+1)L_{n+1}^{k}(x)-(2n+k+1-x)L_{n}^{k}(x)+(n+k)L_{n-1}^{k}(x)=0$$

ii) $$x\frac{d}{dx}L_{n}^{k}(x)-nL_{n}^{k}(x)+(n+k)L_{n-1}^{k}(x)=0.$$

(Solve problem 22.23).

[Warning: You have to watch out the formulas in (1), (2) and (3), which may differ in different books].