Orthogonality and Normalization of Bessel function:

$$\int_0^1 xJ_p(ax)J_p(bx)dx = \begin{cases} 0 & \text{if} \quad a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_p'^2(a)) & \text{if} \quad a = b \end{cases},$$

where *a* and *b* are called zero's of J_p *(x)*.

Proof: **To prove the orthogonality of Bessel function, we have to carry out the followings:**

1. **Rewrite the Bessel function**

   $x^2y'' + xy' + (x^2 - p^2)y = 0$ **in the form**

   $$x(xy')' + (x^2 - p^2)y = 0.$$

2. **Replace *x* by *ax* and you may get** $ax\frac{dy}{dax}$ **equal to** $x\frac{dy}{dx}$ **and similarly** $x(xy')'$ **is unchanged. Thus the DE** $x(xy')' + (a^2x^2 - p^2)y = 0$ **has a solution *y*= J_p *(ax)*.**
3. **Replace *x* by *bx and similarly you may get* the DE** $x(xy')' + (b^2x^2 - p^2)y = 0$ **with its solution *y*= J_p *(bx)*.**
4. **Multiply the DE in (2) by J_p *(bx)* and that in (3) by J_p *(ax)*. Then subtract the resulting equations and divide by x to get:** $\frac{d}{dx}(xJ_p(bx)J_p'(ax) - xJ_p(ax)J_p'(bx)) + (a^2 - b^2)xJ_p(ax)J_p(bx) = 0$
5. **Integrate the resulting equation in (4) from 0 to 1 to obtain:**

   $$(xJ_p(bx)J_p'(ax) - xJ_p(ax)J_p'(bx)\Big|_0^1 + (a^2 - b^2)\int_0^1 xJ_p(ax)J_p(bx)dx = 0$$

6. **The first term of the equation in (5) tends to zero. This is because a) at the lower limit (x=0) J_p *(ax)*, J_p *(bx)*, J'_p *(ax) and* J'_p*(bx) are all finite. b) At the upper limit (x = 1),* J_p *(a) =* J_p *(b) = 0, where a and b are considered as zero's of* J_p.**

Thus the final result is $\boxed{(a^2 - b^2)\int_0^1 xJ_p(ax)J_p(bx)dx = 0}$ **Q.E.D**

Hermite Functions and Laguerre Functions:

These functions arise as solutions of eigenvalue problems in Quantum Mechanics.

Hermite Functions and Ladder operators:

The DE for Hermite functions is $y_n'' - x^2 y_n + (2n+1) y_n = 0$ **(1)**
where *n= 0, 1, 2, 3,*……,etc.
By defining a differential operator $D = \frac{d}{dx}$ **and substituting it into equation (1) we may write the DE in two forms:**

Either

$$(D-x)(D+x)y_n = -2ny_n \quad \textbf{(2)}$$

Or

$$(D-x)(D+x)y_n = -2(n+1)y_n \quad \textbf{(3)}$$

If n is replaced by m in equation (2) and operate with *D+x* on this equation we get:

$$(D+x)(D-x)[(D+x)y_m] = -2m[(D+x)y_m] \quad \textbf{(4)}$$

Also if n is replaced by m in equation (3) and operate on this equation with *D-x*, then we have:

$$(D-x)(D+x)[(D-x)y_m] = -(2m+1)[(D-x)y_m] \quad \textbf{(5)}$$

By comparing equations (2) and (5), you may get:

i) $y_n = [(D-x)y_m]$

and

ii) $n = m+1$

$\therefore y_{m+1} = (D-x)y_m$

. [Here D-x is called raising operator].

Thus we have got a solution y_m of equation (1) for n= m.
How can we get a solution for n= m+1?
Answer:* *We can get the solution for m+1 by just applying the "raising operator" D-x on y_m.

Also by comparing equation (3) and (4), we get

iii) $y_n = [(D+x)y_m]$

and

iv) $n = m-1$

$\therefore y_{m-1} = (D+x)y_m$

. ***[Here D+x is called lowering operator].***

***Notes:* In Q.M. raising and lowering operators are called creation and annihilation operators respectively. And all such operators are called ladder operators.**

Problem: **Use equation (1) to show that** $y_\circ = e^{-x^2/2}$.

Solution: **Put *n* = *0* in equation (1) to get** $y_\circ'' - x^2 y_\circ + y_\circ = 0$.

Also from equation (2) we have $(D-x)(D+x)y_\circ = 0$.

This will give $(\frac{d}{dx} - x)(y_\circ' + xy_\circ) = 0 \Rightarrow y_\circ'' - x^2 y_\circ + y_\circ = 0$.

Thus the operator equation is the same as the original DE.

Now if we find a solution to $(D+x)y_\circ = 0$ **and then operate with *D-x* on this resulting solution we will get the final solution to our DE.**

Thus $(D+x)y_\circ = 0$ **becomes** $\frac{dy}{dx} + xy_\circ = 0$. **This last DE will be simply solved in this way:**

$$\int_0^{y_\circ} \frac{dy_\circ}{y_\circ} = -\int_0^x x dx \Rightarrow \qquad \therefore y_\circ = e^{-x^2/2}.$$

But we have to remember that we are seeking a solution for *n* greater than zero. Here we can operate with (*D-x*) *n times, where* $y_{m+1} = (D-x)y_m$*, remembering that n= m+1.*

$$\therefore y_n = (D-x)^n e^{-x^2/2}$$

These are called the Hermite functions and they can be rewritten as $\therefore y_n = e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}$ **(Solve problem22.3)**

Hermite Polynomials:

We can get the Hermite polynomials simply by developing their Rodrigues formula. This formula can be obtained by multiplying the last equation by $(-1)^n e^{x^2/2}$ **.**

$$\therefore H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$ **(The Rodrigues formula).**

To find the Hermite polynomials ***$H_o(x)$, $H_1(x)$ and $H_2(x)$, using the Rodrigues formula for the Hermite polynomials (problem 22.4), we will have:***

For n = 0 $\Rightarrow$ ***$H_o(x) = 1$***
For n =1 $\Rightarrow$ ***$H_1(x) = 2x$***
For n =2 $\Rightarrow$ ***$H_2(x) = 4x^2 - 2$***

[Note: Suggested problems (22.5, 22.6)]

Properties of Hermite Polynomials:

i) $\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \begin{cases} 0 & if \quad n \neq m \\ \sqrt{\pi}\,2^n n! & if \quad n = m \end{cases}$ ***(Solve problem 10)***

ii) The generating function for the Hermite polynomials can be expressed by $\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty}\frac{h^n}{n!}H_n(x)$

(Solve problem 8).

iii) By using the generating function for the Hermite polynomials, two recursion relations can be found as: (Solve problem 9).

a. $H'_n(x) = 2nH_{n-1}(x)$

b. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$