Bessel's Differential Equation:

Recalling the DE $y'' + n^2 y = 0$ **which has a sinusoidal solution (*i.e.* sin *nx* and cos *nx*) and knowing that these solutions can be treated as power series, we can find a solution to the Bessel's DE which is written as:** $x^2 y'' + xy' + (x^2 - p^2)y = 0$. **The solution is represented by a series. This series very much look like a damped sine or cosine. It is called a Bessel function.**

To solve the Bessel' DE, we apply the Frobenius method by assuming a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^{m+n}$ **.**
Substitute this solution into the DE equation and after some mathematical steps you may find that the indicial equation is $m^2 - p^2 = 0$ **where *m*= ±*p*. Also you may get** $a_1 = 0$ **and hence all odd *a*'s are zero as well. The recursion relation can be found as**

$$a_{n+2} = -\frac{a_n}{(n+m+2)^2 - p^2}$$ **with (*n=0, 1, 2, 3, ……*etc.).**

Case (1): ***m= p (seeking the first solution of Bessel DE)***

We get the solution

$$y = a_{\circ} x^p \left[1 - \frac{x^2}{2(2p+2)} + \frac{x^4}{2 \bullet 4(2p+2)(2p+4)} - \bullet\bullet\bullet \right]$$

This can be rewritten as:

$$J_p(x) = y = a_{\circ} \sum_{n=0}^{\infty} (-1)^n \frac{x^{p+2n}}{2^{2n} n!(p+n)!}$$

Put $a_{\circ} = \frac{1}{2^p p!}$

The Bessel function has the factorial form

$$J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{p+2n}}{n!(p+n)! 2^{2n+p}} ,$$

This is because

$$a_2 = -\frac{a_\circ}{2(2p+2)} = -\frac{a_\circ p!}{2^2(p+1)!}, \qquad a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_\circ p!}{2!2^4(p+2)!},$$

$$a_6 = -\frac{a_4}{6(2p+6)} = -\frac{a_\circ p!}{3!2^6(p+3)!}$$ **and so on.**

Since *(p + n)! = Γ(p + n + 1)*, and *n! = Γ(n + 1)*, *the other form is rewritten as*

$$\Rightarrow \quad J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\Gamma(n+1)\Gamma(p+n+1)} \left(\frac{x}{2}\right)^{2n+p}$$

Case (2): *m= -p* (seeking the second solution of Bessel DE){sec. 13}

The solution in the factorial form is

$$J_{-p}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-p}}{n!(n-p)!2^{2n-p}}$$

Or

$$J_{-p}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\Gamma(n+1)\Gamma(-p+n+1)} \left(\frac{x}{2}\right)^{2n-p}$$

Comments:

***i*) For integer p:**

a. **You can show that *$J_{-p}(x)=(-1)^p J_p(x)$*. Here *$J_{-p}(x)$ is not independent solution.***
b. **It is obvious that we have a problem with *$J_{-p}(x)$ when x=0 because this second solution goes to infinity. While the first solution still exist because it is finite.***
c. **When *p= 2*, the terms in the denominator with *n= 0, 1* go to infinity (because The Gamma of a negative integer is infinity), and these terms do not contribute to the sum. Such case does not exist for a positive p.**

***ii)* For nonintegral *p*:**

$J_{-p}(x)$ and $J_p(x)$ are two independent solutions and a linear combination of them is a general solution. This linear combination is called Neumann function (termed by N_p) or Weber function (termed by Y_p). However this function is

valid for integral or nonintegral p and is also called the Bessel function of second kind:

$$N_p(x) = Y_p(x) = \frac{\cos(\pi p) J_p(x) - J_{-p}(x)}{\sin(\pi p)}$$

Important note:

It must be noted that this expression is an indeterminate form (0/0) for integral p. However for x $\neq$ 0 it has a limit, which is the correct second solution for integral p.

The best general solution may be written as $y = AJ_p(x) + BN_p(x)$, where A and B are arbitrary constants. At x = 0 all N's are $\pm\infty$ and the only solution is the Bessel function of first kind J_p (x).

Some properties of Bessel function:

From the factorial form of J_p (x) you may get:

$$J_\circ(x) = 1 - \frac{x^2}{(1!)^2 2^2} + \frac{x^4}{(2!)^2 2^4} - \frac{x^6}{(3!)^2 2^6} + \bullet\bullet\bullet \quad \textbf{for p= 0}$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{(1!)(2!)2^3} + \frac{x^5}{(2!)(3!)2^5} - \frac{x^7}{(3!)(4!)2^7} + \bullet\bullet\bullet \quad \textbf{for p= 1}$$

From these tow expressions it follows that $\frac{dJ_\circ(x)}{dx} = -J_1(x)$.

Recursion relations for Bessel functions:

1. $$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$$

2. $$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$

3. $$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

4. $$J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x)$$

5. $$J'_p(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$$

Note: **Similar relations also hold for** ***$N_p(x)$*****. [Try to prove such relations].**

Orthogonality and Normalization of Bessel function:

$$\int_0^1 xJ_p(ax)J_p(bx)dx = \begin{cases} 0 & if \quad a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_p'^2(a)) & if \quad a = b \end{cases},$$

where a and b are called zero's of $J_p(x)$.