Legendre Series:

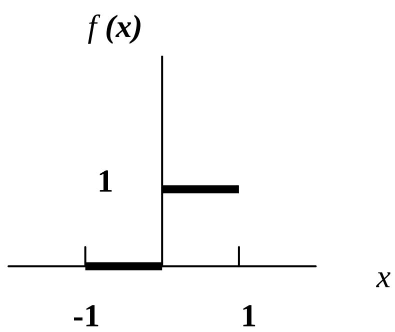
Any function can be expanded in Legendre series as long as the Legendre functions form a complete orthogonal set on (-1, 1), namely,

$$f(x)=\sum_{\ell=0}^{\infty}C_\ell P_\ell(x),$$

where C_ℓ are the coefficients, which are needed to be found.

Example: Expand in a Legendre series the function f (x) given by:

$$f(x)=\begin{cases}0 & -1\langle x\langle 0\\ 1 & 0<x<1\end{cases}$$



Solution: Put $f(x)=\sum_{\ell=0}^{\infty}C_\ell P_\ell(x)$ and find C_ℓ. The latter can be found if we multiply both sides of equation by $P_m(x)$ and integrate from -1 to 1 to get:

$$\int_{-1}^{1}f(x)P_m(x)dx=\sum_{\ell=0}^{\infty}C_\ell\int_{-1}^{1}P_\ell(x)P_m(x)dx=C_m\frac{2}{2m+1}$$

$$\therefore C_m=\frac{\int_{-1}^{1}f(x)P_m(x)dx}{\int_{-1}^{1}[P_\ell(x)]^2dx}$$

Since *f (x) = 0 for -1< x <0 and f (x) = 1 for 0< x <1* then

C_o= 1/2, C_1= 3/4, C_2=0, ……etc. Substitute these values into the above series to get:

$$f(x)=\frac{1}{2}P_\circ(x)+\frac{3}{4}P_1(x)-\frac{7}{10}P_3(x)+\frac{11}{32}P_5(x)-\bullet\bullet\bullet$$

Note: Solve the suggested problems (9.1, 9.6, 9.10)

The associated Legendre functions:

The associated Legendre DE is:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]y=0 \text{ with } m^2\le\ell^2 .$$

The solution is) $P_\ell^m(x)$ it is called the associated Legendre function), where $P_\ell^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$. Using the Rodrigues formula for $P_\ell(x)$ we may rewrite $P_\ell^m(x)$ in the form:

$$P_\ell^m(x)=\frac{1}{2^\ell\,\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

Remarks:

This form of Legendre function is general and its special case is when m= 0. This solution of the DE is for both positive and negative values of m.

For a negative value of m we have $P_\ell^{-m}(x)=(-1)^m\frac{(1-m)!}{(1+m)!}P_\ell^m(x)$,

(see problem 10.8).

For each m, the functions $P_\ell^m(x)$ are a set of orthogonal functions on (-1, 1).(See problem 10.3)

The norm of $P_\ell^m(x)$ can be found as:

$\int_{-1}^{1}[P_\ell^m(x)]^2dx=\frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}$. (See problem 10.10)

The DE of the associated Legendre functions may come in a different form *i.e.* $\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dy}{d\theta})+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]y=0$.

(See problem 10.2).

Suggested problems: (10.2, 10.3, 10.5)