Special Functions

The Factorial and Gamma Functions

The Factorial Function:

$$\int_0^\infty x^n e^{-x} dx = n!$$

n **(=1, 2, 3, ……etc.) is a positive integer.**

Example: **For n=0** $\Rightarrow \int_0^\infty e^{-x} dx = 1 = 0!$

For n=1 $\Rightarrow \int_0^\infty x e^{-x} dx = 1 = 1!$

The Gamma Function:

Two definitions will be considered here.

A. Definition of Gamma functions as integral form.

The Gamma function can be defined in an integral form if we can answer the following question:

Can we define the factorial function for non-integer *n*?

Answer: **Yes, but once we have done that the function will be called Gamma function.**

Now the non-integer "*p*" can replace the integer "*n*" and the Gamma function is defined as:

$$\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx \qquad \text{for } p>0$$

(Here, *the integral converges for p>0*)

Notes:

i. **The integral diverges for** $p \le 0$ **and** $\Gamma(p)$ **can not be defined.**

ii. **For** $0 < p < 1$, x^{p-1} **will go to infinity as the lower limit** $x \to 0$.

The relation between the factorial and Gamma functions:

Take $\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx$

Put p = n to get

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx=(n-1)!$$

Also put *p=n+1* to get

$$\Gamma(n+1)=\int_0^\infty x^{n}e^{-x}dx=n!$$

Examples: $\begin{cases} \Gamma(1) = 0! \\ \Gamma(2) = 1! \\ \Gamma(3) = 2! \\ \Gamma(4) = 3! \end{cases}$

B. Another definition of Gamma function

$\Gamma(p)$ **can be defined as infinite limit and as follows:**

$$\Gamma(p) = \lim_{m\to\infty} \frac{m!\,m^p}{p(p+1)(p+2).........(p+m)}$$

[This limit exists when *p* is a positive integer or zero i.e.

$$\Gamma(p) \to \infty \text{ if } \left.\begin{matrix} p \to 0 \\ p\langle 0 \end{matrix}\right\} so \ \frac{1}{\Gamma(p)} = 0$$ **].**

Now this definition of Gamma function can be tested and compared with the previous definition of Gamma function. For example,

Put p =1 $\Rightarrow \Gamma(1) = \lim_{m\to\infty} \frac{m}{m+1} = 1$**.(Now you can try *p*= 2).**

[Note: A table for the values of Gamma function for *p* between 1 and 2 is available].

What about other positive values of *p* other than 1 and 2?

Answer: The Gamma functions of such values can be

found by a recursion relation.

Recursion relation for the Gamma function:

$$\Gamma(p+1) = p\Gamma(p)$$

[Prove this recursion relation using the two given definitions of Gamma function. The proof will be left to the student as an exercise]

What about the negative values of *p* ?

The Gamma function of negative numbers:

The Gamma function for negative values of *p* can be found using the recursion relation as follows:

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

Examples:

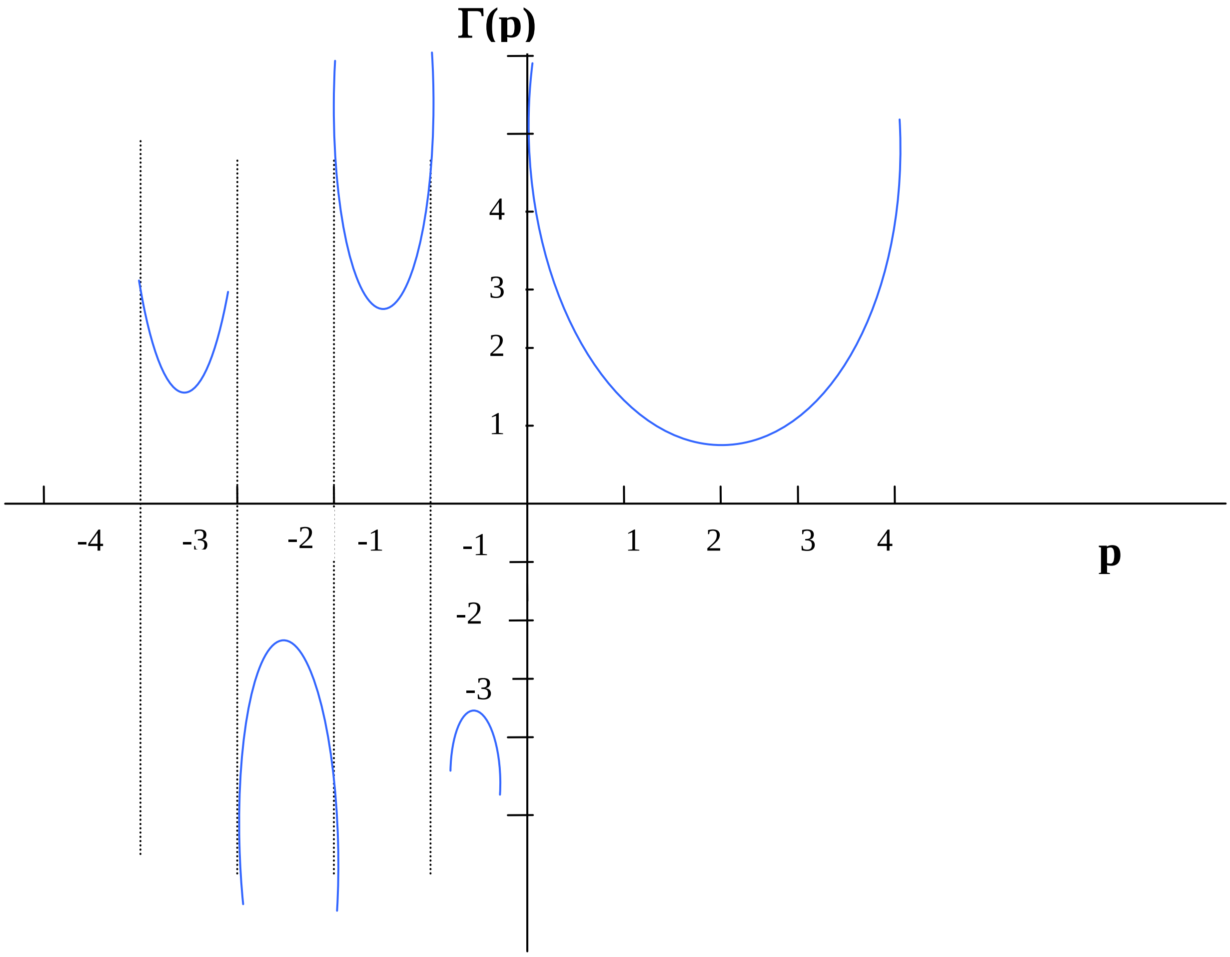
For p = -0.5⇒ $\Gamma(-0.5) = \frac{\Gamma(0.5)}{-0.5}$

For p = -1.5⇒

$$\Gamma(-1.5) = \frac{\Gamma(-0.5)}{-1.5} = \frac{\Gamma(0.5)}{(-1.5)(-0.5)}$$

Note: **Here you can try values of negative integer and zero for p to get** $\Gamma(p) \to \infty$**.**

Figure 1.1: Gamma function for different positive and negative values of *p*.



Conclusions:

i) $\Gamma(p)$ **is a continuous function for all *p*>0**

ii) $\Gamma(p)$ **is a discontinuous function for negative *p*.**

iii) $\Gamma(p)$ **is continuous at intervals between negative integers of *p*, and the values of** $\Gamma(p)$ **alternates from positive to negative at different intervals.**

Some important formulas involving Gamma functions:

(a) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(b) $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin \pi p}$

Note: You can prove the formula in (a), while the other formula in (b) will be proved later in the chapter of complex variables.

Bonus Exercise:

Try to prove the formula in (b). [Hint: Use the identity $\sin p = p\prod_{m=1}^{\infty}(1-\frac{p^2}{m^2\pi^2})$ **which is consistent with the fact that sin** p **has zeros at** p**= 0 and** p**=** $\pm m\pi$**.**

Suggested problems:

(Chapter 11) :

section 3 (3, 4, 5, 13, 17) {Hint: take Γ**(1.7)=0.9086}**

section 5 (1, 2)

section 7 (1, 2, 5, 8)

section 9 (1, 2, 3)