Example: Find the Laurent expansion of $f(z) = \frac{1}{1+z}$ $for|z|>1$ and

for $|z| < 1$.

Solution:

For $|z| < 1$ $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$

For $|z|>1$ = $\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})}$

Replacing z by $-\frac{1}{z}$ in previous example

$$\frac{1}{1+z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3}..........$$

As seen, here, there are different series expansions in different regions of the complex plane.

Example: Consider the Laurent

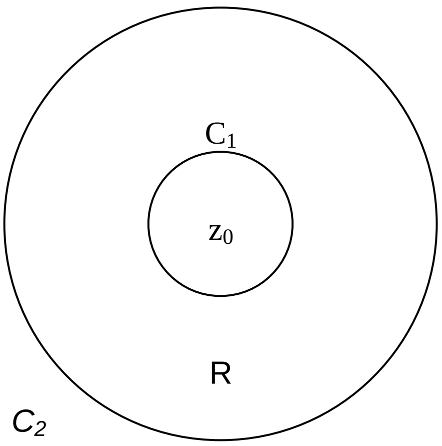
$$f(z) = (1 + (\frac{z}{2}) + (\frac{z^2}{4}) + ...(\frac{z^n}{2})) + \frac{2}{z} + 4(\frac{1}{z^2} - \frac{1}{z^3} + ...\frac{(-1)^n}{z^n} + ...) .$$

Identify the convergence of this series by finding the radii of the circles of convergence.

Solution:

Consider the series of the positive powers (*i.e*. the "a" series). The ratio test tells us that this series converges for $\left|\frac{z}{2}\right| < 1$, that is, for $|z| < 2$. This indicates that this series converges inside a circle C_2 , as shown in the figure, which may be a point.

Applying the ratio test on the series of negative powers (*i.e.* the "b" series) will give a convergence at $\left|\frac{1}{z}\right| < 1$, that is $|z| > 1$. This means that this series converges outside a circle C_1 which may have a radius of infinity.



Example: Given the function $f(z) = \dfrac{12}{z(2-z)(1+z)}$.

a) Identify any possible singular points in this function.
b) Specify the circles of convergence about $z_0 = 0$.
c) Find the Laurent series expansion for the different possible regions.

Solution:

a) There are three singular points: at $z = 0$, $z = 2$ and $z = -1$.
b) There are two circles C_1 and C_2 about $z_0 = 0$. Thus we expect three Laurent series about $z_0 = 0$, namely,

(1) one series in the region R_1 ($0 < |z| < 1$)

(2) another series in the region R_2 ($1 < |z| < 2$) and

(3) the third series in the region ($|z| > 2$).

c) Let us apply the method of partial fractions to the given function.

Rewrite the function as $f(z)=\frac{4}{z}(\frac{3}{(2-z)(1+z)})$

$\therefore \frac{3}{(2-z)(1+z)}=\frac{A}{(2-z)}+\frac{B}{(1+z)}$.

Solve this equation for A and B to get:

A (1 + z) + B (2 - z) = 3

$\Rightarrow$ *A = 1, B = 1*

$$\therefore \; f(z)=\frac{4}{z}(\frac{1}{(2-z)}+\frac{1}{(1+z)})$$

(1) Laurent series for $0\langle|z|\langle 1$

Expand the two terms inside the parentheses as follows:

$$(2-z)^{-1}=\frac{1}{2}(1-\frac{z}{2})^{-1}$$

$$\Rightarrow \quad \frac{1}{2}(1-\frac{z}{2})^{-1}=\frac{1}{2}(1+\frac{z}{2}+\frac{1}{2!}(\frac{z}{2})^{2}+\frac{1}{3!}(\frac{z}{2})^{3}+.....)$$

$$(1+z)^{-1}=1-z+\frac{z^{2}}{2!}-\frac{z^{3}}{3!}+......$$

Substitute the expanded terms into the function $f(z)=\frac{4}{z}(\frac{1}{(2-z)}+\frac{1}{(1+z)})$ to get the required Laurent series in the region $0\langle|z|\langle 1$ as:

$$f(z)=-3+\frac{9z}{4}-\frac{15z^{2}}{24}+.........$$

(2) Laurent series for $1\langle|z|\langle 2$

Expand $(1+z)^{-1} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + ...$ and

$$(1+z)^{-1} = \frac{1}{z(1+\frac{1}{z})}$$

$$\Rightarrow \quad (1+\frac{1}{z})^{-1} = 1 - \frac{1}{z} + (\frac{1}{z})^2 - (\frac{1}{z})^3 + (\frac{1}{z})^4 - .....)$$

Substitute the expanded terms into the function to get:

$$f(z) = \frac{2}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + ... + (\frac{z}{2})^n + 4(\frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z4} - ...\frac{(-1)^n}{z^n})$$

(3) Laurent series for $|z|\rangle 2$

The terms $(1+z)^{-1}$ and $(2-z)^{-1}$ can be rearranged as follows:

$$(1+z)^{-1} = \frac{1}{z}(\frac{1}{1+\frac{1}{z}}) \text{ and } (2-z)^{-1} = -\frac{1}{z}(\frac{1}{1-\frac{2}{z}})$$

Thus by expanding these latter terms to get:

$$(1+\frac{1}{z})^{-1} = 1 - \frac{1}{z} + (\frac{1}{z})^2 - (\frac{1}{z})^3 + ...$$

$$(1-\frac{2}{z})^{-1} = 1 + \frac{2}{z} + (\frac{2}{z})^2 + (\frac{2}{z})^3 + ...$$

Thus

$$f(z) = -\frac{12}{z^3}(1 + \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \frac{11}{z^4} + ...)$$