Contour Integrals:

Theorem 5: Cauchy`s theorem: Let C be a simple closed curve with a continuously turning tangent except possibly at a finite number of points (that is we allow a finite number of corners, but other wise the curve must be "smooth"). If f(z) is analytic on and inside C , then

$\oint f(z)dz = 0$, (this is called a contour integral in the theory of complex variables , which is the line integral in vector analysis)

Proof: (See it in the textbook)

Theorem 6: Cauchy`s integral formula

If a function, $f(z)$, is analytic on and inside a simple curve C, the value of $f(z)$ at a point $z = a$ inside C is given by

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

Proof: (read the proof in the textbook)

Laurent Series:

In many application you may face functions that are generally analytic functions but are not analytic at some points, or in some regions of the complex plane, and consequently, Taylor series cannot be employed in the neighborhood of such points. However, another series representation can frequently be found in which both positive and negative powers of $(z-z_0)$ exist.

Such series is valid for those functions that are analytic in and on a circular annulus $R_1 \le |z-z_0| \le R_2$

Laurent Series formulation:

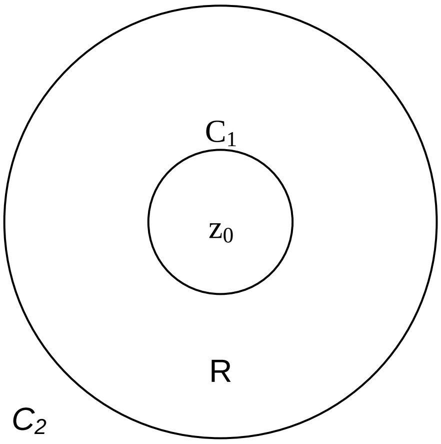
Let C_1 and C_2 be two circles with center at z_0, as shown in the figure below. Let $f(z)$ be analytic in the region "R" between the circles. Then $f(z)$ can be expanded in series of the form,

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\ldots \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots\ldots,$$

OR

$$f(z) = a_0 + \sum_{n=1} a_n(z-z_0)^n + \sum_{n=1} b_n(z-z_0)^{-n}$$, which is convergent in R.

Such a series is called a Laurent series. The "b" series in the above formula is called the principal part of the Laurent series.



Example: Find the Laurent expansion of $f(z) = \frac{1}{1+z}$ $for |z| > 1$ and

for $|z| < 1$.

Solution:

For $|z| < 1$ $\quad \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$

For $|z| > 1$ = $\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})}$

Replacing z by $-\frac{1}{z}$ in previous example

$$\frac{1}{1+z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3}..........$$

As seen, here, there are different series expansions in different regions of the complex plane.

Example: Find the Laurent expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ for

$1 < |z| < 2$.

Solution:

Using the partial fraction decomposition to rewrite $f(z)$ as

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

This can be obtained as follows

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$A(z-2) + B(z-1) = 1$

$(A+B)z - (2A+B) = 1 \qquad \Rightarrow A = -B$

$2A+B = -1$

$A = -1 \qquad B = 1$

Rewriting last equation as:

$$f(z) = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right),$$

Because $1 < |z| < 2$, $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$\Rightarrow f(z) = -\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\left(\frac{1}{z}\right)^n\right) - \frac{1}{2}\left(1 + \left(\frac{z}{2}\right) + \left(\frac{z^2}{4}\right) + ...\left(\frac{z^n}{2}\right)\right).$$

Thus

$$f(z) = \sum_{n=-\infty}^{\infty} C_n Z^n ,$$

where $C_n = \begin{cases} -1 \\ \dfrac{1}{2^{n+1}} \end{cases}$ for $\begin{cases} n \leq -1 \\ n \rangle 0 \end{cases}$

Also, there exist different series expansions for $|z| < 1$ and for $|z| > 2$.