Theorem 2:

If u (x, y) and v (x, y) and their partial derivatives w.r.t x and y are continuous and satisfy the Cauchy–Riemann conditions in a region, then f (z) is analytic at all points inside the region (not necessarily on the boundary).

Proof: (See the proof in the textbook)

Definitions:

- Regular point: It is a point at which f(z) is analytic.
- Singular point or singularity of f(z): It is a point at which f(z) is not analytic. [it is called an isolated singular point if f(z) is analytic every where else inside some small circle about the singular point].

Examples of three types of singularities (at z = 0)

(1) f (z) = $\frac{\sin z}{z}$ (because $\lim_{z \to 0} f$ (z) must exist and f(z) is not analytic)

(2)f (z) = $\frac{1}{\sin z}$ (this is called a pole)

(3)f (z)= $e^{\frac{1}{z}}$ (this is called an essential singularity)

Theorem 3:

If f(z) is analytic in a region R, then it has derivatives of all orders at points inside the region and can be expanded in a Taylor series about any point z_0 inside the region . The power series converges inside the circle about z_0 that extends to the nearest singular point C.

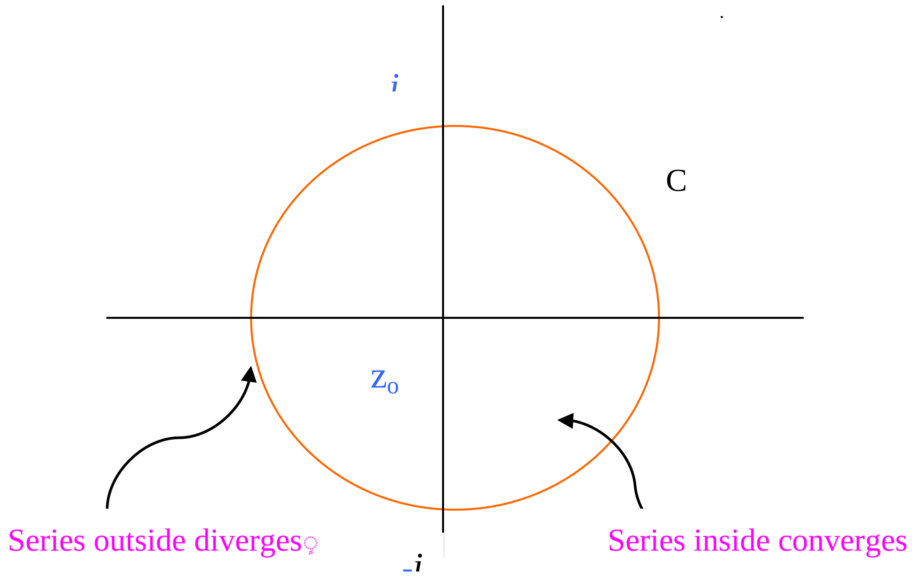
Example: Find the convergence of the function $f(z) = \frac{1}{1+z^2}$ and expand it in a Taylor series at $z = 0$.

Solution:

It must be noted here that $f(z)$, $f'(z)$, $f''(z)$ …….etc, go to infinity at z = $\pm i$. So it is not analytic in any region with z = $\pm i$. Therefore the point z_o of the theorem is the origin and the circle C of convergence of the series extends to the nearest singular point at z= $\pm i$ (*i.e.* - i< z < i or $|z|$ < 1).

Thus f(z) can be expanded in a Taylor series at z_o = 0.

$$f(z) = 1 - z^2 + z^4 - z^4 ....$$



Harmonic function: (like sin ϕ or cos ϕ):

It is the function that satisfies Laplace`s equation $\nabla^2\phi = 0$ or $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ (as represented in x-y plane).

Complex integration: (An introduction to contour integrals)

To consider the evaluation of integrals of complex variable functions along appropriate curves in the complex plane.

(1) The case of complex - valued function *f* of a real variable *t* on a fixed interval $a \leq t \leq b$:

f (t) =u (t) + *i* v (t),

where u(t) and v(t) are real values.

The function f (t) is said to be integrable on the interval [a, b] if the functions *u* and *v* are integrable. Then

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

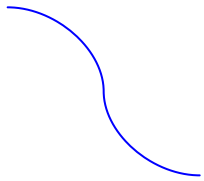
For a continuous functions f(t): $\frac{d}{dt}\int_a^t f(\tau)d\tau = f(t)$ and for $f'(t)$ is continuous when $\int_a^b f'(t)dt = f(b) - f(a)$

(2) The case of complex integration or integration on a curve in the complex plane.

A curve in the complex plane can be described via the parametrization *z (t)* = *x(t)*+ *i y (t)* , in the interval $a \leq t \leq b$.

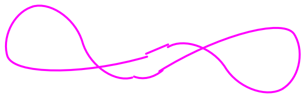
For each given *t* in [a, b] there is a set of points [x (t), y (t)] that are the image points of the interval. The curve is said to be continuous if *x (t)* and *y (t)* are continuous functions of t. Also it is said to be differentiable if x (t) and y (t) are differentiable.

A curve or arc C is a simple one (sometimes called Jordan arc) if it does not intersect itself that is $z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$ for $t \in [a,b]$

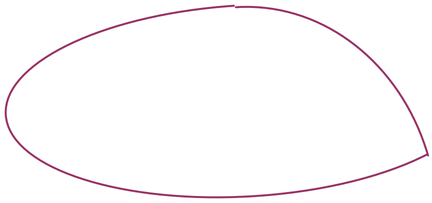


Simple, not closed (Jordan arc)

Not simple, not closed



Not simple closed curve



Simple closed (Jordan curve)

Theorem 4:

Part (1): if f(z) =u + *i*v is analytic in a region , then *u* and *v* satisfy Laplace`s equation in the region (that is , *u* and *v* are harmonic functions)

Part (2): Any function *u*(or *v*) satisfying Laplace`s equation in a simply -connected region , is the real or imaginary part of an analytic function f(z).

•To find the solutions of Laplace`s equation in a complex plane, we need to take the real or imaginary parts of an analytic function of *z*.

Exercise: Take the function u(x, y) $= x^2 - y^2$ which satisfies Laplace`s equation to find f(z).

Since $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ (*u* is a harmonic function)

Firstly, we have to find v(x, y) such that u + *i*v is analytic function of *z*.

Using Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \qquad \text{(integrate w.r.t y)}$$

$\Rightarrow$ v(x, y) = 2xy + g(x)

g(x) is a function of *x* only (which is needed to be found).

Differentiating partially w.r.t. x

$$\frac{\partial v}{\partial x} = 2y + g'(x)$$

But $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$

$\therefore g'(x) = 0 \quad \Rightarrow$ g (x)=const

Then f (z) = u + *i*v = $x^2 - y^2 = 2ixy + const$

u and v are called conjugate harmonic functions.

[*i.e. v* is the harmonic conjugate of *u* (and vice versa)]