Analytic function

In a complex plane where f **(**z**)** is a function **of** complex variable z, f'(z**)** is defined in the same way as derivative of a real function **f'(x**) in a real plane, namely:

$$f'(z) = \frac{df}{dz} = \ell im_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

Definition:

A function **f (z)** is analytic (or holomorphic or regular or monogenic) in a region of the complex plane if it has a (unique) derivative at every point of the region.

[Note: when we say f **(**z**)** is analytic at a point z **=** *a*, it means that f **(**z**)** has a derivative at every point inside some small circle about z = a].

Theorem I:

If f **(**z**) =** *u* **(***x, y***) +** *i* v **(***x, y***)** is analytic in a region, then in that region

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}.$$

(These equations are called Cauchy – Riemann conditions).

Note: The student can prove these equations by using the followings:

(1) **f** has derivative w.r.t **z** ⇒ **f** has partial derivatives w.r.t. x and **y.**

(2) Since a complex function has a derivative w.r.t a real variable if and only of its real and imaginary parts do have derivatives**.**

$$\therefore \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\& \ \frac{\partial f}{\partial z} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})$$

$$\Rightarrow \frac{\partial f}{\partial z} = (\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y})$$

(3) Since we assumed $\frac{\partial f}{\partial z}$ exists and is unique (this is what analytic function all about!!),

Thus $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Example: (Cauchy-Reimann conditions in polar coordinates)

In polar coordinates, prove that:

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial \theta}} \quad \& \quad \boxed{\frac{\partial v}{\partial r} = -\frac{\partial u}{r\partial \theta}}$$

Solution:

In Polar coordinates:

f (z) = u(r, θ)+ i v(r, θ), (1)

where z = r $e^{i\theta}$. (2)

Applying the chain rule to get:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}. \quad \textbf{(3)}$$

Take the partial derivative of eq. (2) w.r.t the variable r to get

$$\frac{\partial z}{\partial r} = e^{i\theta}. \quad \textbf{(4)}$$

Substitute eq. (4) into eq (3) to get

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z} e^{i\theta} . \qquad (5)$$

Take the partial derivative of eq. (1) w.r.t the variable r to get

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} . \qquad (6)$$

From eqs. (5) and (6) we have

$$\frac{\partial f}{\partial z} e^{i\theta} = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} . \qquad (7)$$

Again the chain rule gives us

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial \theta} . \qquad (8)$$

Take the partial derivative of eq. (2) w.r.t the variable θ to get

$$\frac{\partial z}{\partial \theta} = ire^{i\theta} . \qquad (9)$$

Substitute eq. (9) into eq. (8) to get

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial z} ire^{i\theta} . \qquad (10)$$

Take the partial derivative of eq. (1) w.r.t the variable θ to get

$$\frac{\partial f}{\partial \theta} = \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} . \qquad (11)$$

From eqs. (10) and (11) we have

$$\frac{\partial f}{\partial z} e^{i\theta} = \frac{1}{ir} \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) . \qquad (12)$$

From eqs (7) and (12) we have

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = \frac{1}{ir} \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) . \qquad (13)$$

By equating real and imaginary parts in both sides of eq (13) we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial v}{r\partial\theta}} \quad \& \quad \boxed{\frac{\partial v}{\partial r} = -\frac{\partial u}{r\partial\theta}}$$ **Q.E.D**

(These are called Cauchy-Reimann conditions in polar coordinates).

Example: Show that the ***f*** **(z) = e^{z}** is differentiable for all finite values of **z.**

Solution:

e^{z} = e^{x+iy} = e^{x} (cos y + *i* sin y)

u= e^{x} cos y **v=e^{x} sin y**

To show that *f*'(z) = e^{z}:

$\frac{\partial u}{\partial x}$ **= e^{x} cos *y*** , $\frac{\partial v}{\partial y}$ **= e^{x} cos y**

Thus $$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

Also $\frac{\partial u}{\partial y}$ **= - e^{x} sin y** , $\frac{\partial v}{\partial x}$ **= e^{x} sin y**

Hence $$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

⇨ $$f'(z) = e^{x}(\cos y + i\sin y)$$

Thus $f'(z) = e^{x+iy} = e^{z}$

$\therefore$ ***f* (z) = e^{z}** is differentiable for all finite values of z**.**

Exercise: Show that f(z) = e^{iz} is also differentiable for all values of z.

Examples: Given the functions:

i) $|z| = |x + iy|$.

ii) $f(z) = z^2$.

iii) $z^* = x - iy$

a) Find the real and imaginary parts of the given functions.

b) Is each of the given functions analytic? (Use the Cauch-Riemann conditions to check this).

Solutions:

***i*.a)** $u = \sqrt{x^2 + y^2}$ **and v= 0**

***i*.b)** $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$

$\therefore |z|$ **is not analytic.**

(The students must solve the other two examples).