Functions of a complex variable

The complex number, *z,* is simply defined in a complex plane as $z = x + iy$. Here as we know x and y are real numbers. In addition the values of any elementary function of ***z*** such as roots, trigonometric functions and logarithms, ect... can also be found in the complex plane. The complex number *z* can be transformed from Cartesian to Polar coordinate and vice versa in a manner similar to the transformation of any vector in real space, namely,

$$z = x + iy \Leftrightarrow z = r\, e^{i\theta}$$

From now on this z will be considered as the variable of a complex function, namely, f(z).

What are we going to learn?

We are going to learn the followings:

1) the calculus of function f (z) (in particular the differentiation, integration and power series).
2) the basic facts and theorems about functions of a complex variable.

In general: $f(z) = f(x + i\,y)$

OR

$$f(z) = u(x, y) + i\, v(x, y)$$

This means that the complex function of z **can also be defined in terms of two real function** *u(x, y)* and *v (x, y)* **which are, in turn, functions of the real variables *x* and *y*.**

Example: Take $f(z) = z^2$ and use the definition of $z = x + i\,y$ to find the two functions *u(x, y)* and *v (x, y)*.

Solution:

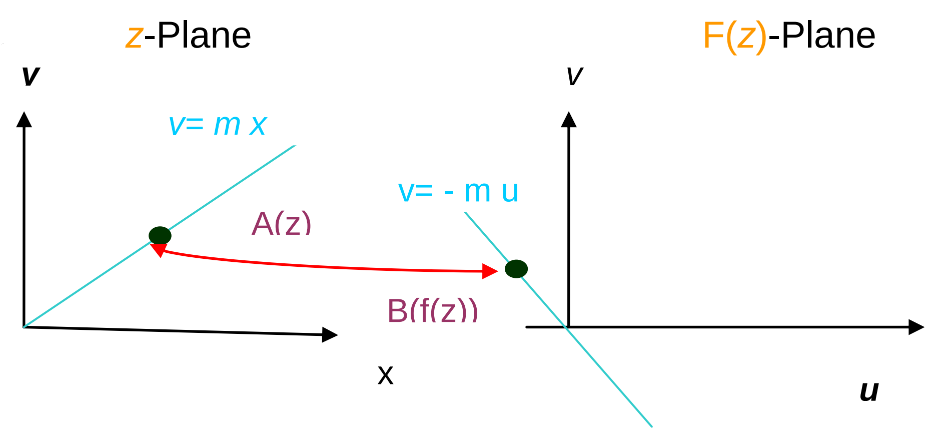
$$f(z) = (x + i\,y)^2 = x^2 - y^2 - 2i\,x\,y.$$

Here we get ⇨ $u\,(x, y) = x^2 - y^2$

⇨ $v\,(x, y) = -2i\,x\,y$

Transformations

Every point in the z-plane has just one point corresponds to it in the f (z)-plane. Also every point in the f (z)-plane corresponds to just one point in *t*he z-plane. This is said to be one-to-one transformation.



This one-to-one transformation can be explained by the following example.

Example:

If **f (z) =** $u + i\,v$ and $z = x + i\,y$ **are** related by f **(z) =** $\frac{1}{z}$

(as in the previous plot), knowing that $y = m\,x$**,**

show that $v = -\,m\,u$.

Solution:

Since f (z) = $\frac{1}{z}$ **thus** $u + iv = \frac{1}{x + iy}$

$$\Rightarrow \qquad = \frac{x - iy}{x^2 + y^2}$$

$$\Rightarrow \qquad u = \frac{x}{x^2 + y^2} \quad \textbf{and} \quad v = -\frac{y}{x^2 + y^2}.$$

Putting $y = mx$ **in previous expressions to get:**

$$u = \frac{1}{x(1+m^2)} \quad \& \quad v = -\frac{m}{x(1+m^2)}.$$

By eliminating x**, we get** $v = -\,m\,u$

A classification of a complex function:

Complex function
- Single – valued function
- Multiple – valued function

Single–valued function means that **f (*z*)** has only one (complex) value in each ***z****.*

Multiple-valued function means that it has more than a value for each z. Each value is called a branch. So this complex function has several branches.

[Warning: we need to be very careful when we deal with multiple-valued function]

Examples on finding the real and imaginary parts *u* **(***x***,** *y*) and v **(***x***,** *y*) of complex functions.

Examples:

1) $|z|$

2) $z^* = x - iy$

3) $\cosh z$

4) $\frac{2z - i}{iz + 2}$

Solutions:

1) $|z| = |x + iy| = (x^2 + y^2)^{1/2}$ ⇨ $u = (x^2 + y^2)^{1/2}$ and $v = 0$.

2) $z^* = x - iy$ ⇨ $u = x$ and $v = -y$.

3) $\cosh z = \cosh(x + iy)$

$= \cosh x \cosh iy + \sinh x \sinh iy$

But $\cosh iy = (e^{iy} + e^{-iy})/2 = \cos y$

Also $\sinh iy = (e^{iy} - e^{-iy})/2i = i(e^{+iy} + e^{-iy})/2i = i \sin y$

⇨ $\cosh z = \cos y \cosh x + i \sin y \sinh x$.

Thus ⇨ $u = \cos y \cosh x$ and $v = \sin y \sinh x$.

[Note: Example 4 will be left as an exercise]