General Curvilinear Coordinates

Put x_1, x_2, x_3 as the set of variables or coordinates, and

h_1, h_2, h_3 as the cooresponding scale factors.

Here,

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2 = \sum_{i=1}^{3} h_i^2 dx_i^2 \tag{1}$$

In rectangular coordinates:

$$x_1 = x \qquad x_2 = y \qquad x_3 = z \tag{2}$$

$$h_1 = 1 \qquad h_2 = 1 \qquad h_3 = 1$$

In cylindrical coordinates:

$$\begin{aligned} &x_1 = r \qquad x_2 = \theta \qquad x_3 = z \\ &h_1 = 1 \qquad h_2 = r \qquad h_3 = 1 \end{aligned} \tag{3}$$

In general, the vector displacement $\vec{ds}$ can be written as:

$$\vec{ds} = \hat{e}_1 h_1 dx_1 + \hat{e}_2 h_2 dx_2 + \hat{e}_3 h_3 dx_3 = \sum_{i=1}^{3} \hat{e}_i h_i dx_i \tag{4}$$

Also, the volume element:

$$dV = h_1 h_2 h_3 dx_1 dx_2 dx_3 \,, \tag{5}$$

in an orthogonal system.

e.g. In rectangular coordinate system $\Rightarrow \; dV = dxdydz$.

In cylindrical coordinate system $\Rightarrow \; dV = rdrd\theta dz$.

If the coordinate system is not orthogonal:

$$\begin{aligned} ds^2 = {} & g_{11} dx_1^2 + g_{12} dx_1 dx_2 + g_{13} dx_1 dx_3 \\ & + g_{21} dx_2 dx_1 + g_{22} dx_2^2 + g_{23} dx_2 dx_3 \\ & + g_{31} dx_3 dx_1 + g_{32} dx_3 dx_2 + g_{33} dx_3^2 \end{aligned} \tag{6}$$

or ,

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} dx_i dx_j \qquad (7)$$

Or, alternatively, in a matrix form:

$$ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \qquad (8)$$

If the system is orthogonal then,

$$ds^2 = g_{11} dx_1^2 + g_{22} dx_2^2 + g_{33} dx_3^2 \qquad (9)$$

(Here g_{ij} represents a diagonal matrix)

The scale factors
$$g_{11} = h_1^2 \qquad g_{22} = h_2^2 \qquad g_{33} = h_3^2 \qquad (10)$$

$g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} = 0$ (for an orthogonal coordinate system only).

Vector operations in orthogonal curvilinear coordinates

The gradient is defined as $\vec{\nabla}$ u:

In rectangular coordinate systems:

$$\vec{\nabla} u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z} \qquad (11)$$

In cylindrical coordinates:

$$\vec{\nabla} u = \hat{e}_r \frac{\partial u}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial u}{\partial \theta} + \hat{e}_z \frac{\partial u}{\partial z} \qquad (12)$$

(If r changes while θ and z are kept constants$\Rightarrow$ $ds = dr$, and the r component of $\vec{\nabla} u \Rightarrow \frac{du}{ds}$; where $ds = dr$, that is $\frac{\partial u}{\partial r}$).

(Also if θ changes when r and z are kept fixed, then the component of $\vec{\nabla} u \Rightarrow \frac{du}{ds}$ where $ds = r d\theta \Rightarrow \frac{1}{r}\frac{\partial u}{\partial \theta}$).

In general orthogonal coordinats x_1, x_2, x_3, the component of $\vec{\nabla} u$ in the x_1 direction is $\frac{du}{ds}$ if $ds = h_1 dx_1$; that is the component of $\vec{\nabla} u$ in the direction of $\hat{e}_1$ is $\frac{1}{h_1}\left(\frac{\partial u}{\partial x_1}\right)$.

$$\overrightarrow{\nabla u} = \hat{e}_1 \frac{1}{h_1}\frac{\partial u}{\partial x_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial u}{\partial x_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial u}{\partial x_3} = \sum_{i=1}^{3} \frac{\hat{e}_i}{h_i}\frac{\partial u}{\partial x_i}. \tag{13}$$

Divergence: $\vec{\nabla} \bullet \vec{V}$

The general relation for the divergence of a vector $\vec{V}$ is

$$\vec{\nabla} \bullet \vec{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 V_1)}{\partial x_1} + \frac{\partial(h_1 h_3 V_2)}{\partial x_2} + \frac{\partial(h_1 h_2 V_3)}{\partial x_3}\right] \tag{14}$$

The proof of equation 14 is as follows:

Let us define the vector

$$\vec{V} = \hat{e}_1 V_1 + \hat{e}_2 V_2 + \hat{e}_3 V_3, \tag{15}$$

in an orthogonal system .

Rewrite equation 15 as

$$\vec{V} = \frac{\hat{e}_1}{h_2 h_3}(h_2 h_3 V_1) + \frac{\hat{e}_2}{h_1 h_3}(h_1 h_3 V_2) + \frac{\hat{e}_3}{h_1 h_2}(h_1 h_2 V_3). \tag{16}$$

Take the divergence of both sides of equation 16 to get,

$$\vec{\nabla} \bullet \vec{V} = \vec{\nabla} \bullet \left(\frac{\hat{e}_1}{h_2 h_3}\right)(h_2 h_3 V_1) + \vec{\nabla} \bullet \left(\frac{\hat{e}_2}{h_1 h_3}\right)(h_1 h_3 V_2) + \vec{\nabla} \bullet \left(\frac{\hat{e}_3}{h_1 h_2}\right)(h_1 h_2 V_3). \tag{17}$$

Put $\phi = h_2 h_3 V_1$ for $\vec{v} = \frac{\hat{e}_1}{h_2 h_3}$ and apply the identity

$$\vec{\nabla} \bullet (\phi \vec{v}) = \vec{v} \bullet (\vec{\nabla} \phi) + \phi \vec{\nabla} \bullet \vec{v} \tag{18}$$

The 1st term on the right-hand-side of equation 17 will give:

$$\vec{\nabla}\bullet\left(h_1h_3V_1\frac{\hat{e}_1}{h_2h_3}\right)=\frac{\hat{e}_1}{h_2h_3}\bullet\vec{\nabla}\left(h_2h_3V_1\right)+h_2h_3V_1\vec{\nabla}\bullet\left(\frac{\hat{e}_1}{h_2h_3}\right)$$

$$=\frac{\hat{e}_1}{h_2h_3}\bullet\vec{\nabla}\left(h_2h_3V_1\right)$$

$$=\frac{\hat{e}_1}{h_2h_3}\bullet\left(\frac{\hat{e}_1}{h_1}\frac{\partial h_2h_3V_1}{\partial x_1}+\frac{\hat{e}_2}{h_2}\frac{\partial h_2h_3V_1}{\partial x_2}+\frac{\hat{e}_3}{h_3}\frac{\partial h_2h_3V_1}{\partial x_3}\right) \qquad (19)$$

$$=\frac{1}{h_2h_3h_1}\frac{\partial h_2h_3V_1}{\partial x_1}.$$

Similarly,

$$\vec{\nabla}\bullet\left(h_1h_3V_2\frac{\hat{e}_2}{h_1h_3}\right)=\frac{1}{h_1h_3h_2}\frac{\partial\left(h_1h_3V_2\right)}{\partial x_2} \qquad (20)$$

Also ,

$$\vec{\nabla}\bullet\left(h_1h_2V_3\frac{\hat{e}_3}{h_1h_2}\right)=\frac{1}{h_1h_2h_3}\frac{\partial\left(h_1h_2V_3\right)}{\partial x_3}, \qquad (21)$$

$$\therefore\ \vec{\nabla}\bullet\overline{V}=\frac{1}{h_1h_2h_3}\left[\frac{\partial\left(h_2h_3V_1\right)}{\partial x_1}+\frac{\partial\left(h_1h_3V_2\right)}{\partial x_2}+\frac{\partial\left(h_1h_2V_3\right)}{\partial x_3}\right].\ \text{(Q.E.D)} \qquad (22)$$

Where the following relations are substituted into equations 19 and 17:

$$\vec{\nabla}\bullet\left(\frac{\hat{e}_3}{h_1h_2}\right)=0$$

$$\vec{\nabla}\bullet\left(\frac{\hat{e}_2}{h_1h_3}\right)=0 \qquad (23)$$

$$\vec{\nabla}\bullet\left(\frac{\hat{e}_1}{h_2h_3}\right)=0$$

Such relations can be proved as follows:

Use equation 13 and put $u=x_1$ to get

$$\vec{\nabla}x_1 = \frac{\hat{e}_1}{h_1}. \qquad (24)$$

Similarly, put $u = x_2$ in same equation to get

$$\vec{\nabla}x_2 = \frac{\hat{e}_2}{h_2}. \qquad (25)$$

Again put $u = x_3$ in same equation to get

$$\vec{\nabla}x_3 = \frac{\hat{e}_3}{h_3}. \qquad (26)$$

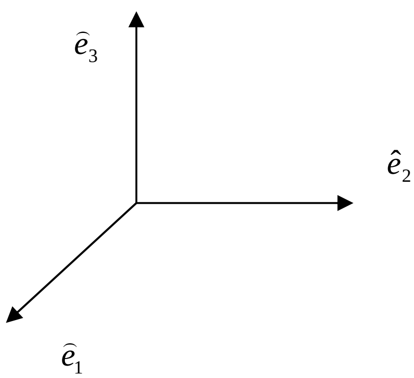
Take $\vec{\nabla}x_1 \times \vec{\nabla}x_2 = \dfrac{\hat{e}_1 \times \hat{e}_2}{h_1 h_2}$ (27)

$\because \quad \hat{e}_1 \times \hat{e}_2 = \hat{e}_3$ for orthogonal system (as shown in the figure),

$$\therefore \vec{\nabla}x_1 \times \vec{\nabla}x_2 = \frac{\hat{e}_3}{h_1 h_2}. \qquad (28)$$

Thus, the divegence of equation 28 gives

$$\vec{\nabla} \bullet \vec{\nabla}x_1 \times \vec{\nabla}x_2 = 0 \qquad (29)$$



From equations 28 and 29 we get

$$\vec{\nabla} \bullet \frac{\hat{e}_3}{h_1 h_2} = 0. \qquad \text{(Q.E.D)} \qquad (30)$$

The other divergence relations, in equation 23, can be proved in the same way.

In cylindrical coordinates:

Use equation 14 and the following equation (31)

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = 1$$
$$x_1 = r, \qquad x_2 = \theta, \qquad x_3 = z, \qquad (31)$$

to get:

$$\vec{\nabla} \bullet \vec{V} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_r) + \frac{\partial}{\partial \theta}V_\theta + \frac{\partial}{\partial z}(rV_z)\right], \qquad (32)$$

Or $\Rightarrow$ $$\vec{\nabla} \bullet \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}. \qquad (33)$$

Laplacian: $(\nabla^2 u)$

The general relation is:

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right]. \qquad (34)$$

In cylindrical coorinates:

Substitute equation 31 into 34 to get,

$$\nabla^2 u = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial u}{\partial z}\right)\right], \qquad (35)$$

or,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (36)$$

The curl $(\vec{\nabla} \times \vec{V})$:

The general relation for the curl of a vector $\vec{V}$ is

$$\vec{\nabla}\times\vec{V}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix} \qquad (37)$$

Expanding the determinent in equation 37 will give

$$=\frac{\hat{e}_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3V_3)-\frac{\partial}{\partial x_3}(h_2V_2)\right]+\frac{\hat{e}_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1V_1)-\frac{\partial}{\partial x_1}(h_3V_3)\right]+\frac{\hat{e}_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2V_2)-\frac{\partial}{\partial x_2}(h_1V_1)\right]$$

In cylindrical coordinates:

From equations 31 and 37 we get

$$\vec{\nabla}\times\vec{V}=\frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}, \qquad (38)$$

or,

$$\vec{\nabla}\times\vec{V}=\hat{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial\theta}-\frac{\partial V_\theta}{\partial z}\right)+\hat{e}_\theta\left(\frac{\partial V_r}{\partial z}-\frac{\partial V_z}{\partial r}\right)+\frac{1}{r}\hat{e}_2\left(\frac{\partial}{\partial r}(rV_\theta)-\frac{\partial V_r}{\partial\theta}\right). \qquad (39)$$

Spherical Polar Coordinates:

From equation 9, we can easily show that ds^2 can be written as

$$ds^2=dx^2+dy^2+dz^2 \qquad (40)$$

However, in spherical polar coordinates, ds^2 can be shown to be

$$ds^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \qquad (41)$$

Proof:

We know

$$x=r\sin\theta\cos\phi,\ \ y=r\sin\theta\sin\phi,\ \ z=r\cos\theta. \qquad (42)$$

Take the derivative of first relation in equation 42, to get

$$dx=-r\sin\theta\ \sin\phi\ \ d\phi+\cos\phi(r\cos\theta\ \ d\theta+\sin\theta\ \ dr)$$

$$dx=\sin\theta\ \ \cos\phi\ \ dr+r\cos\theta\ \ \cos\phi\ \ d\theta-r\sin\theta\ \ \sin\phi\ \ d\phi \qquad (43)$$

Squaring equation 43 gives

$$dx^2 = (\sin\theta \ \cos\phi \ dr + r\cos\theta \ \cos\phi \ d\theta)^2 + r^2\sin^2\theta \ \sin^2\phi \ d\phi^2$$

$$-2r\sin\theta \ \sin\phi \ d\phi \ \left(\sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta\right)$$

$$dx^2 = \sin\theta^2\cos\phi^2 dr^2 + r^2\cos^2\theta\cos^2\phi d\theta^2 + r^2\sin\theta^2\sin^2\phi d\phi^2$$

$$+2r\sin\theta \ \cos\theta\cos^2\phi \ d\theta \ dr - 2r^2\sin\theta^2\sin\phi\cos\phi \ drd\phi$$

$$-2r^2\sin\theta\cos\theta\sin\phi\cos\phi \ d\theta \ d\phi \qquad (44)$$

Similarly, we get

$$dy = r\sin\theta\cos\phi \ d\phi + \sin\phi(r\cos\theta \ d\theta + \sin\theta \ dr)$$

$$dy = r\sin\theta\cos\phi \ d\phi + r\cos\theta\sin\phi \ d\theta + \sin\phi\sin\theta \ dr, \qquad (45)$$

and,

$$dy^2 = r^2\sin^2\theta\cos^2\phi d\phi^2 + (r\cos\theta\sin\phi \ d\theta + \sin\phi\sin\theta \ dr)^2$$

$$+2r\sin\theta\cos\phi \ d\phi(r\cos\theta\sin\phi \ d\theta + \sin\phi\sin\theta \ dr)$$

$$dy^2 = r^2\sin^2\theta\cos^2\phi d\phi^2 + r^2\cos^2\theta\sin^2\phi d\theta^2 + \sin^2\phi\sin^2\theta dr^2$$

$$+2r\sin\theta \ \cos\theta\sin^2\phi \ d\theta \ dr + 2r^2\sin\theta\cos\theta\sin\phi\cos\phi \ d\phi \ d\theta$$

$$+2r\sin^2\theta\sin\phi\cos\phi \ dr \ d\phi. \qquad (46)$$

Also,

$$dz = -r\sin\theta \ d\theta + \cos\theta \ dr, \qquad (47)$$

and,

$$dz^2 = r^2\sin^2\theta d\theta^2 + \cos^2\theta dr^2 - 2r\sin\theta\cos\theta \ drd\theta \qquad (48)$$

Using equation 4 and 46 to get

$$dx^2 + dy^2 = \sin^2\theta(\sin^2\phi + \cos^2\phi)dr^2 + r^2\cos^2\theta d\theta^2(\sin^2\phi + \cos^2\phi)$$

$$+r^2\sin^2\theta d\phi^2(\sin^2\phi + \cos^2\phi) + 2r\sin\theta \ \cos\theta \ d\theta \ dr \qquad (49)$$

From equation 48 and 49, we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \text{ (Q.E.D)} \qquad (50)$$

The vector $\overrightarrow{ds}$ in spherical polar coordinates can be writtn as

$$\overrightarrow{ds} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r\sin\theta \ d\phi \qquad (51)$$

This equation (51) is obtained when the following scale factors are substiuted into equation 4.

The scale factors: $$h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta \qquad (52)$$

Also the volume element dV can be easily obtained from equations 5 and 52, namely,

$$dV = r^2 \sin\theta \; dr \; d\theta \; d\phi \qquad (53)$$

The relation between $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ of spherical coordinats and $\hat{i}, \hat{j}, \hat{k}$ of Cartesian coordinates systems:

Starting with

$$\vec{ds} = \hat{i}\,dx + \hat{j}dy + \hat{k}dz \qquad (54)$$

$$= \hat{i}\,(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi) + \hat{j}(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi) + \hat{k}(\frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta) \qquad (55)$$

Using equations 42 and 55 we get

$$\vec{ds} = \hat{i}\,(\sin\theta\,\cos\phi\;dr + r\cos\theta\cos\phi\;d\theta - r\sin\theta\sin\phi\;d\phi)$$
$$+\,\hat{j}(\sin\theta\sin\phi\;dr + r\cos\theta\sin\phi\;d\theta + r\sin\theta\cos\phi\;d\phi)$$
$$+\,\hat{k}(\cos\theta\;dr - r\sin\theta\;d\theta\;)$$

$$\vec{ds} = (\hat{i}\,\sin\theta\cos\phi + \hat{j}\,\sin\theta\sin\phi + \hat{k}\,\cos\theta)\;dr$$
$$+\,(\hat{i}\,\cos\theta\cos\phi + \hat{j}\,\cos\theta\sin\phi - \hat{j}\,\sin\theta)rd\theta$$
$$+\,(-\hat{i}\,\sin\phi + \hat{j}\,\cos\phi + 0\hat{k})r\sin\theta\;d\phi \qquad (56)$$

Thus we have

$$\therefore \hat{e}_r = \hat{i}\,\sin\theta\cos\phi + \hat{j}\,\sin\theta\sin\phi + \hat{k}\,\cos\theta \qquad (57)$$

$$\hat{e}_\theta = \hat{i}\,\cos\theta\cos\phi + \hat{j}\,\cos\theta\sin\phi - \hat{k}\,\sin\theta \qquad (58)$$

$$\hat{e}_\phi = -\hat{i}\sin\phi + \hat{j}\cos\phi\,. \tag{59}$$

Vector Operations in Spherical Coordinates:

In spherical polar coordinates, we have

$$\begin{array}{lll} \hat{e}_1 = \hat{e}_r & \hat{e}_2 = \hat{e}_\theta & \hat{e}_3 = \hat{e}_\phi \\ h_1 = 1 & h_2 = r & h_3 = r\sin\theta \\ x_1 = r & x_2 = \theta & x_3 = \phi \end{array} \tag{60}$$

From equation s 13 and 60 we may have

$$\vec{\nabla}u = \hat{e}_r\frac{\partial u}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial u}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} \tag{61}$$

Also from 14 and 60 we get

$$\vec{\nabla}\bullet\vec{V} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta V_r\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta V_\theta\right) + \frac{\partial}{\partial\phi}\left(rV_\phi\right)\right]. \tag{62}$$

Using equations 34 and 60 gives us

$$\nabla^2 u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)\right]. \tag{63}$$

Equation 37 and 60 may result in

$$\vec{\nabla}\times\vec{V} = \frac{\hat{e}_1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3V_3) - \frac{\partial}{\partial x_3}(h_2V_2)\right] + \frac{\hat{e}_2}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1V_1) - \frac{\partial}{\partial x_1}(h_3V_3)\right]$$
$$+\frac{\hat{e}_3}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2V_2) - \frac{\partial}{\partial x_2}(h_1V_1)\right]$$

$$\vec{\nabla}\times\vec{V} = \frac{\hat{e}_r}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\sin\theta V_\phi\right) - \frac{\partial}{\partial\phi}\left(rV_\theta\right)\right] + \frac{\hat{e}_\theta}{r\sin\theta}\left[\frac{\partial}{\partial\phi}V_r - \frac{\partial}{\partial r}\left(r\sin\theta V_\phi\right)\right]$$
$$+\frac{\hat{e}_\phi}{r}\left[\frac{\partial}{\partial r}\left(rV_\theta\right) - \frac{\partial}{\partial\theta}V_r\right]. \tag{64}$$