Graduate Stat. Mech

HW # 3 - Solution

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① Pathria 3.6

a) $S = -k_B \sum_r P_r \ln P_r$; to maximize S that is subject to the constraint $\sum_r P_r = 1$, we introduce a Lagrange multiplier λ and extremize the quantity

$$S' = S - \lambda \left(\sum_r P_r - 1 \right)$$

$$= -k_B \sum_r P_r \ln P_r - \lambda \left(\sum_r P_r - 1 \right)$$

$$\frac{dS'}{dP_r} = 0 \Rightarrow -k_B \sum_r \left(P_r \frac{1}{P_r} + \ln P_r \right) - \lambda \left(\sum_r 1 \right) = 0$$

$$-k_B \left(\sum_r 1 + \sum_r \ln P_r \right) - \lambda \sum_r 1 = 0$$

$$\Rightarrow \sum_r \left(-k_B - k_B \ln P_r - \lambda \right) = 0 \Rightarrow -k_B (1 + \ln P_r) - \lambda = 0$$

$$\Rightarrow 1 + \ln P_r = -\frac{\lambda}{k_B} \Rightarrow \ln P_r = -\frac{\lambda}{k_B} - 1 \Rightarrow P_r = e^{-\frac{\lambda}{k_B} - 1} = \text{const}$$

so P_r is independent of r and according to the constraint

$$\sum_r P_r = 1 \Rightarrow \sum_r e^{-\frac{\lambda}{k_B} - 1} = e^{-\frac{\lambda}{k_B} - 1} \underbrace{\sum_r}_{\Omega} = 1 \Rightarrow P_r \Omega = 1$$

$\Rightarrow P_r = \frac{1}{\Omega}$, so all states are equally likely to occur.

b) Now, we have ~~one~~ more constraint ; $\sum_r E_r P_r \equiv E$

let us introduce γ as a second lagrange multiplier and then we extremize

$$s' = -k_B \sum_r P_r \ln P_r - \lambda\left(\sum_r P_r - 1\right) - \gamma\left(\sum_r E_r P_r - E\right)$$

$$\frac{ds'}{dP_r} = -k_B\left(\sum_r 1 + \sum_r \ln P_r\right) - \lambda \sum_r 1 - \gamma\left(\sum_r E_r\right) = 0$$

$$\Rightarrow \sum_r \left(-k_B - k_B \ln P_r - \lambda - \gamma E_r\right) = 0$$

$$-k_B - k_B \ln P_r - \lambda - \gamma E_r = 0 \Rightarrow -k_B(1 + \ln P_r) - \lambda - \gamma E_r = 0$$

$$\Rightarrow 1 + \ln P_r = \frac{-\lambda - \gamma E_r}{k_B} \Rightarrow \ln P_r = \frac{-\lambda - \gamma E_r}{k_B} - 1 = \frac{-\lambda - \gamma E_r - k_B}{k_B}$$

$$\Rightarrow P_r = e^{-\frac{\lambda}{k_B} - \frac{\gamma E_r}{k_B} - 1} = \underbrace{e^{-\frac{\lambda}{k_B} - 1}}_{\text{constant}} e^{-\frac{\gamma E_r}{k_B}} = C e^{-\frac{\gamma E_r}{k_B}} = C e^{-\beta E_r}$$

where $\beta = \frac{\gamma}{k_B}$

β to be determined by the given value of $\bar{E}$

c) Here we add a third constraint $\langle N \rangle = \bar{N} = \sum_r N_r P_r \equiv N$

$$s' = -k_B \sum_r P_r \ln P_r - \lambda\left(\sum_r P_r - 1\right) - \gamma\left(\sum_r E_r P_r - E\right) - \rho\left(\sum_r E_r N_r - N\right)$$

$$\frac{ds'}{dP_r} = -k_B(\ln P_r + 1) - \lambda - \gamma E_r - \rho N_r = 0$$

$$\Rightarrow 1 + \ln P_r = \frac{-\lambda - \gamma E_r - \rho N_r}{k_B} \Rightarrow$$

$$\Rightarrow \ln P_r = \frac{-\lambda - \gamma E_r - \rho N_r}{k_B} - 1$$

$$\ln P_r = -\frac{\lambda}{k_B} - \frac{\gamma E_r}{k_B} - \frac{\rho N_r}{k_B} - 1$$

$$\Rightarrow P_r = \underbrace{e^{-\frac{\lambda}{k_B} - 1}}_{\text{constant}} \, e^{-\frac{\gamma E_r}{k_B} - \frac{\rho N_r}{k_B}}$$

$$= C e^{-\beta E_r - \mu N_r} \quad \text{--- (1)} \quad ; \quad \mu = \frac{\rho}{k_B}$$

where μ to be determined from the given value of $\bar{N} \equiv N$

- Note that from (1), if we remove the constraint imposed by $\bar{N}$, then $\mu = 0$ and hence

$$P_r = C e^{-\beta E_r}$$

furthermore, if we remove the constraint imposed by $\bar{E}$, then $\beta = 0$ and hence

P_r = constant recovering the microcanonical ensemble, where all states are equally probable.

② $N_+ + N_- = N$

$$+\varepsilon \text{ ——— } N_+$$
$$-\varepsilon \text{ ——— } N_-$$

a) total energy

$$E = N_+ \varepsilon + N_-(-\varepsilon) = (N_+ - N_-)\varepsilon = (2N_+ - N)\varepsilon$$

$$\Rightarrow N_+ = \frac{1}{2}\left(N + \frac{E}{\varepsilon}\right) \quad \text{and } N_- = N - N_+ = N - \frac{1}{2}N - \frac{E}{2\varepsilon}$$

$$= \frac{1}{2}\left(N - \frac{E}{\varepsilon}\right)$$

so $\Omega = \binom{N}{N_+} = \dfrac{N!}{(N-N_+)!\,N_+}$ or $\Omega = \binom{N}{N_-} = \dfrac{N!}{(N-N_-)!\,N_-!} = \dfrac{N!}{N_+!\,N_-!}$

$$= \frac{N!}{N_-!\,N_+!}$$

now $\dfrac{S}{k_B} = \ln \Omega = \ln N! - \ln N_+! - \ln N_-!$; using $\ln n! = n\ln n - n$ we have

$$= N\ln N - \cancel{N} - N_+ \ln N_+ + \cancel{N_+} - N_- \ln N_- + \cancel{N_-}$$

$$= (N_+ + N_-)\ln N - N_+ \ln N_+ - N_- \ln N_-$$

$$= N_+ \ln N + N_- \ln N - N_+ \ln N_+ - N_- \ln N_-$$

$$= N_+ \ln \frac{N}{N_+} + N_- \ln \frac{N}{N_-} = -N_+ \ln \frac{N_+}{N} - N_- \ln \frac{N_-}{N} \quad \text{--- (1)}$$

but $\dfrac{N_+}{N} = \dfrac{N}{2N} + \dfrac{E}{2N\varepsilon} = \dfrac{N\varepsilon}{2N\varepsilon} + \dfrac{E}{2N\varepsilon} = \dfrac{N\varepsilon + E}{2N\varepsilon}$

and $\dfrac{N_-}{N} = \dfrac{N\varepsilon - E}{2N\varepsilon}$

so eq^n 1 becomes

$$\frac{S}{k_B N} = -\frac{N\varepsilon + E}{2N\varepsilon} \ln\left(\frac{N\varepsilon + E}{2N\varepsilon}\right) - \frac{N\varepsilon - E}{2N\varepsilon} \ln\left(\frac{N\varepsilon - E}{2N\varepsilon}\right)$$

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b) we have $S = k_B N \left[-\frac{N_+}{N} \ln \frac{N_+}{N} - \frac{N_-}{N} \ln \frac{N_-}{N} \right]$

let $x = \frac{N_+}{N}$ and using $N_- = N - N_+ \Rightarrow \frac{N_-}{N} = 1 - \frac{N_+}{N} = (1-x)$

$\therefore \quad S = k_B N \left[-x \ln x - (1-x) \ln (1-x) \right]$

Now $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \left(\frac{\partial S}{\partial x}\right)_N \left(\frac{\partial x}{\partial E}\right)_N$; where

$$x = \frac{N_+}{N} = \frac{N\varepsilon + E}{2N\varepsilon} = \frac{1}{2} + \frac{E}{2N\varepsilon} \; ; \quad \left(\frac{\partial x}{\partial E}\right)_N = \frac{1}{2N\varepsilon}$$

and $\left(\frac{\partial S}{\partial x}\right)_N = k_B N \left[-\cancel{x} \cdot \frac{1}{\cancel{x}} - \ln x - \cancel{(1-x)} \left(\frac{-1}{\cancel{(1-x)}}\right) + \ln (1-x) \right]$

$$= k_B N \left\{ \cancel{-1} - \ln x \cancel{+1} + \ln (1-x) \right\} = k_B N \ln \left(\frac{1-x}{x}\right)$$

$\Rightarrow \frac{1}{T} = k_B \cancel{N} \ln \left(\frac{1-x}{x}\right) \cdot \frac{1}{2\cancel{N}\varepsilon} = \frac{k_B}{2\varepsilon} \ln \left(\frac{1-x}{x}\right)$

$\Rightarrow \frac{2\varepsilon}{k_B T} = \ln \left(\frac{1-x}{x}\right) \Rightarrow \frac{1-x}{x} = e^{2\varepsilon/k_B T} \Rightarrow \frac{1}{x} - 1 = e^{2\varepsilon/k_B T}$

$\Rightarrow \frac{1}{x} = e^{2\varepsilon/k_B T} + 1 \Rightarrow x = \frac{1}{1+e^{2\varepsilon/k_B T}} = \frac{N_+}{N}$

$$\Rightarrow \boxed{N_+ = \frac{N}{1+e^{2\varepsilon/k_B T}}}$$

similarly $\frac{N_-}{N} = 1 - x = 1 - \frac{1}{1+e^{2\varepsilon/k_B T}} = \frac{\cancel{1} + e^{2\varepsilon/k_B T} \cancel{-1}}{1+e^{2\varepsilon/k_B T}} = \frac{e^{2\varepsilon/k_B T}}{1+e^{2\varepsilon/k_B T}}$

$\Rightarrow \frac{N_-}{N} = \frac{1}{e^{-2\varepsilon/k_B T} + 1} \Rightarrow \boxed{N_- = \frac{N}{1+e^{-2\varepsilon/k_B T}}}$

c) Canonical ensemble $Q_N = \sum_E g(E)\, e^{-\beta E}$

↳ multiplicity (degeneracy)

here, we have only two states, so we can not convert the summation into integral

E goes from $-N\varepsilon \longrightarrow N\varepsilon$

$-N\varepsilon$: all particles in state $-\varepsilon$

$N\varepsilon$: all particles in state $+\varepsilon$

so $g(E) = g(N_+) = \binom{N}{N_+} = \dfrac{N!}{N_+!\,N_-!} = g(N_-) = \binom{N}{N_-}$

$$\Rightarrow Q = \sum_{N_+=0}^{N} g(N_+)\, e^{-\beta(N_+\varepsilon - N_-\varepsilon)} \quad \text{or} \quad \sum_{N_-=0}^{N} g(N_-)\, e^{-\beta(N_+\varepsilon - N_-\varepsilon)}$$

↓ go with this

$$= \sum_{N_+} g(N_+)\, e^{-\beta(N_+\varepsilon - (N-N_+)\varepsilon)}$$

$$= \sum_{N_+} g(N_+)\, e^{-\beta(2N_+ - N)\varepsilon}$$

$$= e^{N\beta\varepsilon} \underbrace{\sum_{N_+=0}^{N} \binom{N}{N_+} e^{-\beta 2N_+\varepsilon}}_{\text{Binomial series}} \quad ; \text{ using } \sum_{k=0}^{n} \binom{n}{k} r^k = (1+r)^n$$

$$= e^{N\beta\varepsilon} \sum_{N_+=0}^{N} \binom{N}{N_+} \left(e^{-2\beta\varepsilon}\right)^{N_+} = e^{N\beta\varepsilon}\left(1+e^{-2\beta\varepsilon}\right)^N$$

$$= \left(e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right)^N = Q_1^N \quad ; \text{ where } Q_1 = e^{\beta\varepsilon} + e^{-\beta\varepsilon}$$

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d)

at $T=0$

$x \to 0$, $N_+ \to 0$, $N_- \to N$; $E \to -N\varepsilon$, $S=0$

all particles in the lower level, so there is one unique microstate ($\Omega=1$) and hence entropy vanishes ($S = k_B \ln \Omega = 0$)

at $T \to \infty$, $x \to 1/2$, $N_+ \to \frac{N}{2}$ and $N_- \to \frac{N}{2}$,

$E \to 0$, $S \longrightarrow N k_B \ln 2$

the two energy levels are equally occupied. This state has the greatest degrees of freedom, and hence maximum entropy at $x = 1/2$

③ Pathria 3.7

we have shown that $C_P - C_V = \dfrac{\alpha^2 V T}{K_T} > 0$ if $K_T \geq 0$

it is known for many substances that $K_T \geq 0 \Rightarrow C_P - C_V > 0$

now $\left(\dfrac{\partial P}{\partial T}\right)_V \left(\dfrac{\partial T}{\partial V}\right)_P \left(\dfrac{\partial V}{\partial P}\right)_T = -1 \Rightarrow \left(\dfrac{\partial P}{\partial T}\right)_V = -\dfrac{1}{\left(\dfrac{\partial T}{\partial V}\right)_P \left(\dfrac{\partial V}{\partial P}\right)_T}$

where $\alpha = \dfrac{1}{V}\left(\dfrac{\partial V}{\partial T}\right)_P$, $K_T = -\dfrac{1}{V}\left(\dfrac{\partial V}{\partial P}\right)_T$

$$= -\frac{\left(\frac{\partial V}{\partial T}\right)_P}{\left(\frac{\partial V}{\partial P}\right)_T} = \frac{\alpha}{K_T}$$

$$\Rightarrow C_P - C_V = \frac{K_T^2 \left(\frac{\partial P}{\partial T}\right)^2 V T}{K_T} = K_T \left(\frac{\partial P}{\partial T}\right)_V^2 V T$$

$$= -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)^2 V T = -T \left(\frac{\partial P}{\partial T}\right)_V^2 \Big/ \left(\frac{\partial P}{\partial V}\right)_T$$

now using $A = -k_B T \ln Q$

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T} = k_B T \left(\frac{\partial \ln Q}{\partial V}\right)_T$$

$$\Rightarrow C_P - C_V = -T \frac{\left[\frac{\partial}{\partial T}\left(k_B T \left(\frac{\partial \ln Q}{\partial V}\right)_T\right)_V\right]^2}{\frac{\partial}{\partial V}\left(k_B T \left(\frac{\partial \ln Q}{\partial V}\right)_T\right)_T} = -k_B \frac{\left[\frac{\partial}{\partial T}\left(T\left(\frac{\partial \ln Q}{\partial V}\right)_T\right)_V\right]^2}{\left(\frac{\partial^2 \ln Q}{\partial V^2}\right)_T} > 0$$

- for an ideal gas

$PV = N k_B T$

$P = k_B T \left(\dfrac{\partial \ln Q}{\partial V}\right)_T$

$\dfrac{P}{k_B T} = \left(\dfrac{\partial \ln Q}{\partial V}\right)_T$

$\dfrac{N}{V} = \left(\dfrac{\partial \ln Q}{\partial V}\right)_T$

$\dfrac{\partial}{\partial V}\left(\dfrac{N}{V}\right) = \left(\dfrac{\partial^2 \ln Q}{\partial V^2}\right)_T$

$-\dfrac{N}{V^2} = \left(\dfrac{\partial^2 \ln Q}{\partial V^2}\right)_T$

$$\Rightarrow C_P - C_V = -k_B \frac{\left[\left(\frac{\partial \ln Q}{\partial V}\right)_T + T \underbrace{\frac{\partial}{\partial T}\left(\frac{N}{V}\right)}_{0}\right]^2}{-N/V^2} = \frac{-k_B \left(\frac{N}{V}\right)^2}{-N/V^2}$$

$= +k_B N$ as expected

④ pathria 3.15

Extreme relativistic gas of N monoatomic particles with energy relation $\varepsilon = pc$; c is speed of light

$$Q_N(V,T) = \frac{1}{N!}\left[Q_1(V,T)\right]^N \; ; \; Q_1(V,T) = \int_0^\infty e^{-\beta\varepsilon} g(\varepsilon)\, d\varepsilon$$

when $g(\varepsilon)$ is the single particle density of states which is given by

$$g(\varepsilon) = \frac{1}{h^3}\int d^3q\, d^3p\, \delta(\varepsilon - \varepsilon_p) \; ; \; \varepsilon_p = pc \; ; \; d\varepsilon_p = c\,dp$$

$$= \frac{4\pi V}{h^3}\int_0^\infty p^2 dp\, \delta(\varepsilon-\varepsilon_p) = \frac{4\pi V}{h^3}\int_0^\infty \frac{\varepsilon_p^2}{c^2}\frac{d\varepsilon_p}{c}\,\delta(\varepsilon-\varepsilon_p)$$

$$= \frac{4\pi V}{(hc)^3}\int_0^\infty d\varepsilon_p\, \varepsilon_p^2\, \delta(\varepsilon-\varepsilon_p) = \frac{4\pi V}{(hc)^3}\varepsilon^2$$

Now $$Q_1(N,T) = \int_0^\infty e^{-\beta\varepsilon}\frac{4\pi V}{(hc)^3}\varepsilon^2 d\varepsilon = \frac{4\pi V}{(hc)^3}\int_0^\infty d\varepsilon\, \varepsilon^2 e^{-\beta\varepsilon}$$

$$= \frac{4\pi V}{(hc)^3}\frac{2}{\beta^3} \; ; \text{ when I used } \int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$$

$$= \frac{4\pi V}{(hc\beta)^3} \implies Q_N = \frac{1}{N!}\left[8\pi V\left(\frac{k_BT}{hc}\right)^3\right]^N$$

$$\implies A = -k_BT \ln Q_N = -k_BT \ln\left[\frac{1}{N!}\left(8\pi V\left(\frac{k_BT}{hc}\right)^3\right)^N\right]$$

$$= -k_BT\left\{N \ln 8\pi V\left(\frac{k_BT}{hc}\right)^3 - (N\ln N - N)\right\}$$

Now using A, we find

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T} = k_B T N \frac{\frac{8\pi}{N}\left(\frac{k_B T}{hc}\right)^3}{\frac{8\pi V}{N}\left(\frac{k_B T}{hc}\right)^3} = \frac{k_B T N}{V}$$

$$\Rightarrow PV = N k_B T \quad \text{as expected}$$

$$E = U = k_B T^2 \left(\frac{\partial \ln Q_N}{\partial T}\right)_{V,N} \quad ; \quad \ln Q_N = N \ln \frac{8\pi V}{N}\left(\frac{k_B T}{hc}\right)^3 + N$$

$$= k_B T^2 \left[N \frac{\frac{8\pi V}{N} 3\left(\frac{k_B T}{hc}\right)^2 \left(\frac{k_B}{hc}\right)}{\frac{8\pi V}{N}\left(\frac{k_B T}{hc}\right)^3} \right] = k_B T^2 \left\{ 3N \left(\frac{k_B}{hc}\right) \frac{hc}{k_B T} \right\}$$

$$= k_B T^2 \frac{3N}{T}$$

$$= 3N k_B T \quad \text{as expected}$$

Now $C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3N k_B$

$$C_P = \left(\frac{\partial H}{\partial T}\right)_P = \left(\frac{\partial}{\partial T}(E + PV)\right)_P = \frac{\partial}{\partial T}(3N k_B T + N k_B T)$$

$$= \frac{\partial}{\partial T}(4N k_B T) = 4N k_B$$

$$\gamma = \frac{C_P}{C_V} = \frac{4N k_B}{3N k_B} = \frac{4}{3} \quad \text{as expected}$$

⑤ ideal gas of N particles with $H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2} k r_i^2$

a)

$$Q_N = \frac{Q_1^N}{N!} \; ; \; Q_1 = \frac{1}{h^2} \int d^2r \, d^2p \, e^{-\beta H} \; ; \; d^2r = r\,dr\,d\theta$$

$$d^2p = p\,dp\,d\phi$$

$$Q_1 = \frac{1}{h^2} \int r\,dr\,d\theta \, e^{-\beta \frac{kr^2}{2}} \cdot \int p\,dp\,d\phi \, e^{-\frac{\beta p^2}{2m}} \; ; \quad \begin{array}{l} 0 \leq r \leq R \\ 0 \leq p < \infty \\ 0 \leq \theta, \phi \leq 2\pi \end{array}$$

$$= \frac{2\pi}{h^2} \int_0^R r\,dr \, e^{-\beta \frac{kr^2}{2}} \cdot 2\pi \underbrace{\int_0^\infty p\,dp \, e^{\frac{-\beta p^2}{2m}}}_{\frac{1}{2\left(\frac{\beta}{2m}\right)} = \frac{m}{\beta}}$$

$$= \frac{4\pi^2}{h^2} \frac{m}{\beta} \int_0^R r\,dr \, e^{-\frac{\beta k r^2}{2}}$$

let $x = \frac{\beta k r^2}{2}$; $dx = \beta k \, r\,dr$

$$Q_1 = \frac{4\pi^2 m}{h^2 \beta} \frac{1}{\beta k_\beta} \int_0^{\frac{\beta k r^2}{2}} dx \, e^{-x} = -\frac{4\pi^2 m}{h^2 \beta^2 k_\beta} \left[e^{-x} \right]_0^{\frac{\beta k r^2}{2}}$$

$$= \frac{-4\pi^2 m}{h^2 \beta^2 k_\beta} \left[e^{-\frac{\beta k R^2}{2}} - 1 \right] = \frac{4\pi^2 m}{h^2 \beta^2 k_\beta} \left[1 - e^{-\frac{\beta k R^2}{2}} \right]$$

$$Q_N = \frac{1}{N!} \left(\frac{4\pi^2 m}{h^2 \beta^2 k_\beta} \right)^N \left[1 - e^{-\frac{\beta k R^2}{2}} \right]^N$$

b)

$$E = -\frac{\partial \ln Q_N}{\partial \beta} \; ;$$

where $\ln Q_N = N \ln \frac{4\pi^2 m}{h^2 \beta^2 k_\beta} + N \ln \left(1 - e^{-\beta \frac{kR^2}{2}}\right) - \underbrace{\ln N!}_{\text{constant w.r.t } T, V}$

$$\Rightarrow E = -\left[\frac{N \frac{4\pi^2 m}{h^2 k_\beta}\left(\frac{-2\beta}{\beta^4}\right)}{\frac{4\pi^2 m}{h^2 \beta^2 k_\beta}} + N \frac{\left(\frac{kR^2}{2} e^{-\beta\frac{kR^2}{2}}\right)}{1-e^{-\beta\frac{kR^2}{2}}} + 0 \right]$$

$$= \frac{2N}{\beta} - \frac{N \frac{kR^2}{2} e^{-\beta\frac{kR^2}{2}}}{1-e^{-\beta\frac{kR^2}{2}}} = \frac{2N}{\beta} - \frac{NkR^2/2}{e^{\beta\frac{kR^2}{2}}-1}$$

Note that if $k \to 0$ (~~N~~o interactions), then
$\Downarrow$ very small

$$e^{\beta\frac{kR^2}{2}} \approx 1 + \beta\frac{kR^2}{2} \Rightarrow E \approx \frac{2N}{\beta} - \frac{N}{\beta} \approx \frac{N}{\beta} = Nk_B T$$

which is the same result of the equipartition theorem with 2N degrees of freedoms ($\frac{1}{2} f k_B T$)

- Now in 2D, the pressure is defined by

$$P = -\left(\frac{\partial A}{\partial a}\right)_{T,N} \quad ; \quad a \text{ is the area} \qquad a = \pi R^2$$

$$da = 2\pi R\, dR$$

$$\frac{da}{dR} = 2\pi R$$

$$= -\left(\frac{\partial A}{\partial R}\right)\left(\frac{\partial R}{\partial a}\right)$$

$$= -\frac{\partial A}{\partial R} \cdot \frac{1}{2\pi R} \quad ; \text{ where}$$

$$A = -k_B T \ln Q_N = -k_B T \left[N \ln\left(\frac{4\pi^2 m}{h^2 \beta^2 k_B}\right) + N \ln\left(1-e^{-\beta\frac{kR^2}{2}}\right) - \ln N! \right]$$

$$\frac{\partial A}{\partial R} = -k_B T \left\{ N \frac{\beta k R\, e^{-\beta\frac{kR^2}{2}}}{1-e^{-\beta kR^2/2}} \right\}$$

$$\frac{\partial A}{\partial R} = \frac{-NkR\, e^{-\beta \frac{kR^2}{2}}}{1 - e^{-\beta\frac{kR^2}{2}}} = -NkR \Big/ \left(e^{\beta\frac{kR^2}{2}} - 1\right)$$

$$\Rightarrow P = \frac{1}{2\pi R} \cdot \frac{NkR}{e^{\beta\frac{kR^2}{2}} - 1} = \frac{Nk}{2\pi} \frac{1}{e^{\beta\frac{kR^2}{2}} - 1}$$

Note that if $k \to 0$, $e^{\beta\frac{kR^2}{2}} \approx 1 + \beta\frac{kR^2}{2}$

$$\Rightarrow P = \frac{Nk}{2\pi} \frac{2}{\beta k R^2} = \frac{N}{\pi R^2} k_B T = \frac{N}{a} k_B T$$

$$\Rightarrow \boxed{Pa = Nk_B T}$$ as expected in 2D

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{\beta}{T}\left[\frac{2N}{\beta^2} - N\left(\frac{kR^2}{2}\right)^2 \frac{e^{\beta kR^2/2}}{\left(e^{\beta kR^2/2} - 1\right)^2}\right]$$

again if $k \to 0$

$$\frac{e^{\beta kR^2/2}}{\left(e^{\beta kR^2/2} - 1\right)^2} \approx \frac{1}{(\beta kR^2/2)^2} \approx \frac{1}{\beta^2\left(\frac{kR^2}{2}\right)^2}$$

$$\Rightarrow C_V = \frac{\beta}{T}\left[\frac{2N}{\beta^2} - N\cancel{\left(\frac{kR^2}{2}\right)^2} \frac{1}{\beta^2\cancel{\left(\frac{kR^2}{2}\right)^2}}\right]$$

$$= \frac{\beta}{T}\left[\frac{2N}{\beta^2} - \frac{N}{\beta^2}\right]$$

$$= \frac{\beta}{T}\frac{N}{\beta^2} = \frac{N}{T\frac{1}{k_B T}} = Nk_B$$ as expected

c) probability $= \dfrac{q_{ab}}{q_1}$

$$= \frac{\int_a^b e^{-\beta \frac{kR^2}{2}} R\,dR}{\int_0^R e^{-\beta kR^2/2} R\,dR} \longrightarrow$$

let $x = \beta k R^2/2$
$dx = \beta k R\,dR$
$R\,dR = dx/\beta k$

$$= \frac{e^{-\beta k \frac{a^2}{2}} - e^{-\beta k \frac{b^2}{2}}}{1 - e^{-\beta k R^2/2}}$$

Note that if $a \longrightarrow 0$ and $b \longrightarrow R$

$\Rightarrow P = 1$

⑥ pathria 3.31 a Fermi oscillator has an energy eigenvalues of $\varepsilon_n = n\varepsilon$; $n = 0, 1$

$$Q_1(\beta) = \sum_{n=0}^{1} e^{-\beta n \varepsilon} = 1 + e^{-\beta\varepsilon}$$

$Q_N = Q_1^N$ classical counting as done in Sec 3.8 Pathria

$$= (1 + e^{-\beta\varepsilon})^N$$

$$\Rightarrow A = -k_B T \ln Q_N = -k_B T N \ln(1 + e^{-\beta\varepsilon})$$

Now $\mu = \left(\frac{\partial A}{\partial N}\right)_{T,V} = -k_B T \ln(1 + e^{-\beta\varepsilon})$

$$P = -\left(\frac{\partial A}{\partial V}\right)_{T,N} = 0$$

$$S = -\left(\frac{\partial A}{\partial T}\right)_{V,N} = N k_B \ln(1 + e^{-\beta\varepsilon}) + \frac{N\varepsilon e^{-\beta\varepsilon}}{T(1 + e^{-\beta\varepsilon})}$$

$$\Rightarrow E = A + TS = -k_B T N \cancel{\ln(1 + e^{-\beta\varepsilon})} + TN\cancel{k_B \ln(1 + e^{-\beta\varepsilon})} + \frac{N\varepsilon e^{-\beta\varepsilon}}{1 + e^{-\beta\varepsilon}}$$

$$= \frac{N\varepsilon e^{-\beta\varepsilon}}{1 + e^{-\beta\varepsilon}}$$

$$= \frac{N\varepsilon}{e^{\beta\varepsilon} + 1}$$

$$\Rightarrow \frac{E}{N} = \frac{\varepsilon}{1 + e^{\beta\varepsilon}}$$